Variance reduction with Normalizing Flows

Rene Poncelet

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THE HENRYK NIEWODNICZAŃSKI
INSTITUTE OF NUCLEAR PHYSICS
POLISH ACADEMY OF SCIENCES

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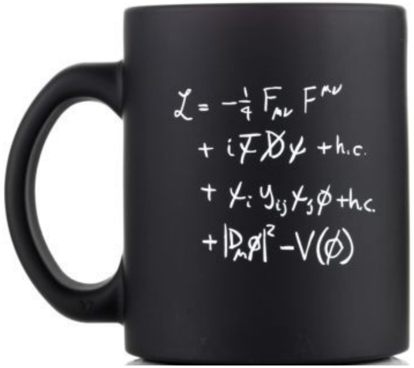
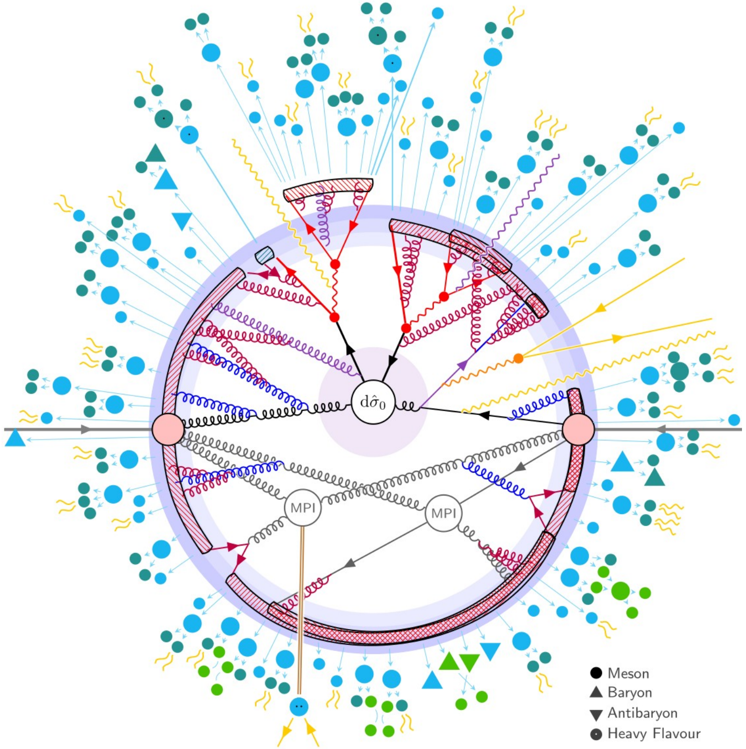
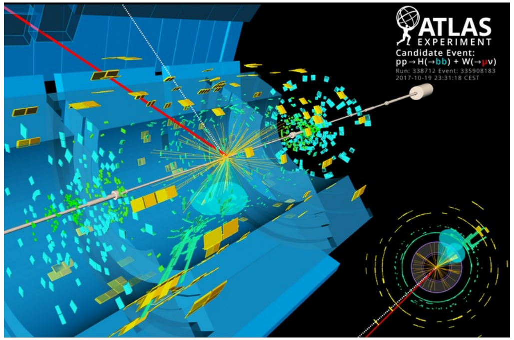
Monte Carlo Event Simulations

… the **bridge** between “theory world” and “real world”



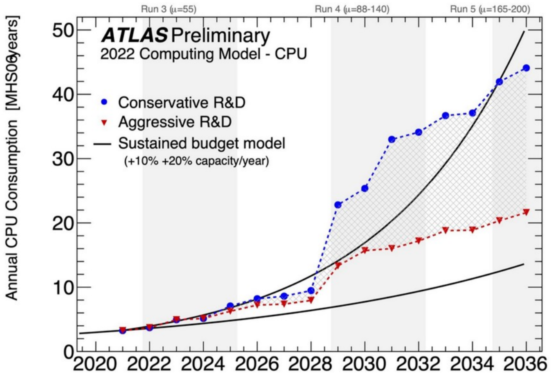


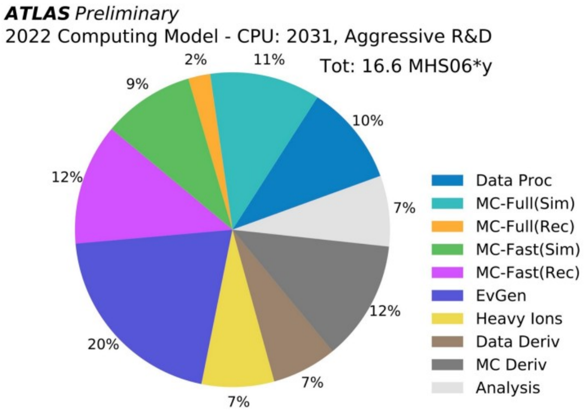
[Pythia 8.3]



The HL-LHC Computing Problem

[CERN-LHCC-2022-005]





Generating events is/will be computationally expensive

1) **negative events**

→ decrease in effective sample size

→ cumbersome down stream in analysis etc.

2) **expensive higher point ME unweighting**

bites twice:

→ higher-point matrix elements are expensive

→ complex phase space reduces sample efficiency

The HL-LHC Computing Problem

Generating events is/will be computationally expensive
1) **negative events**
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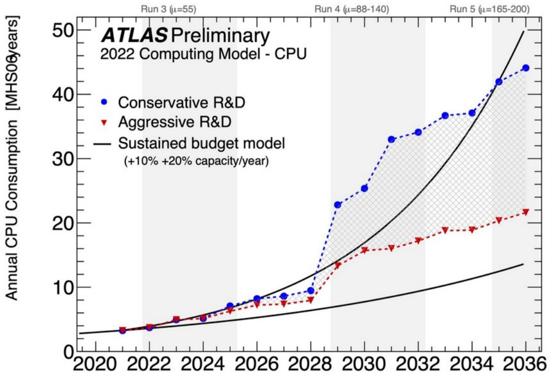
[CERN-LHCC-2022-005]



Exploit modern architectures → CPU + GPU

- MEs: PEPPER [Bothmann, Childers, Giele, Höche, Isaacson Knobbe'23]
- Madgraph on GPUs [Valassi, Childers, Hageböck, Massaro, Mattelaer, Nichols, Optolowicz, Roiser, Teig, Wettersten'25]

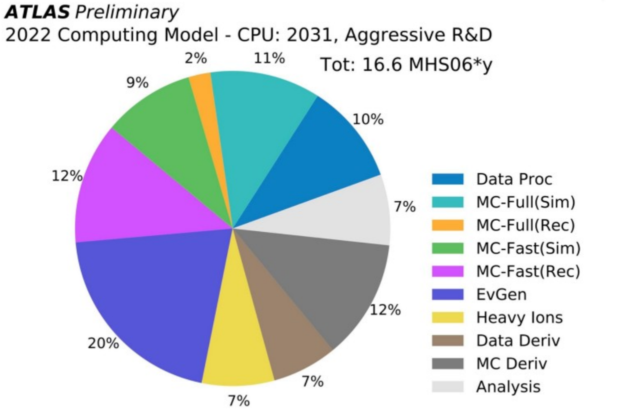
Ensure positivity of event weights

- Positive matching: ESME [PanScales'25], KrK [Pratixan, Siodmok, Whitehead '24]
- Reweighting methods: cell-resampling [J. Andersen, A. Maier'21+Maître'23] Arcane [Shyamsundar'25]

Machine Learning Methods

[De Crescenzo, Mariño, Elmer, Heimel, Plehn, Winterhalder, Zaro '26][Bothmann, Janßen, Knobbe, Schmitzer, Sinz'26][Jansen, Poncelet, Schumann '25]

- Neural Importance Sampling
- Surrogates to emulating MEs



Theory picture of hadron collision events

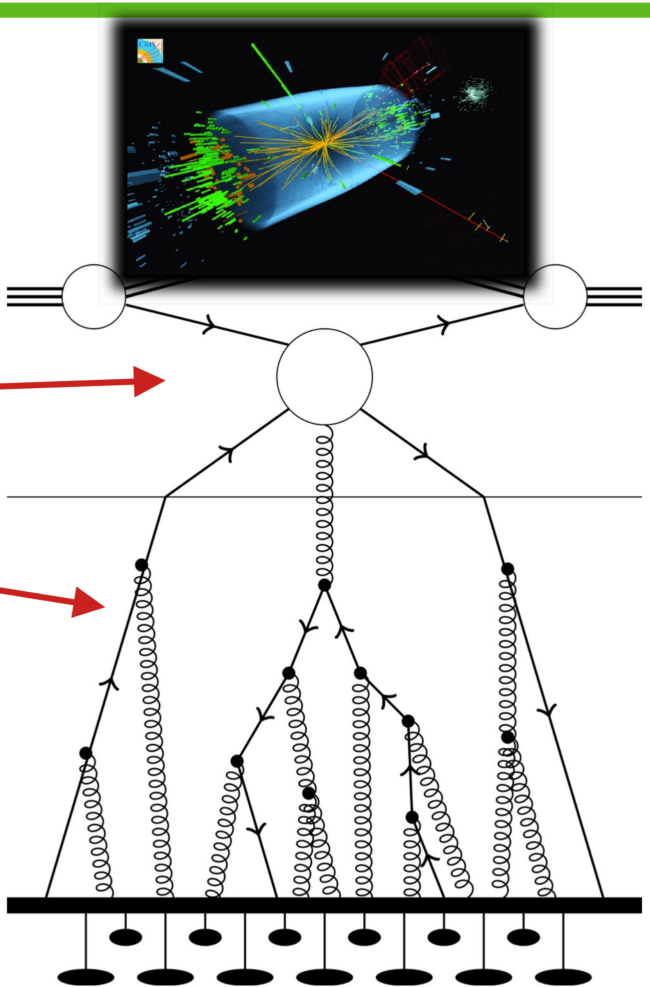
Guiding principle: factorization
"What you see depends on the energy scale"

In Quantum Chromodynamics (QCD):

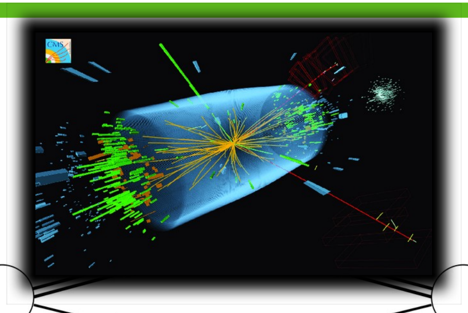
$Q \gg \Lambda_{\mathrm{QCD}}$ **Fixed-order perturbation theory**
scattering of individual partons

$Q \gtrsim \Lambda_{\mathrm{QCD}}$ **Parton-shower/Resummation**
all-order bridge between perturbative and non-perturbative physics

$Q \sim \Lambda_{\mathrm{QCD}}$ **"Hadronization"/MPI/...**
non-perturbative physics



Theory picture of hadron collision events



Guiding principle: factorization
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$Q \sim \Lambda_{\mathrm{QCD}}$	**"Hadronization"/MPI/…** non-perturbative physics

Focus here:

- Higher-order cross sections
- "Unweighted" (matched) events are the seeds of parton-shower evolution

Neural importance sampling & surrogates

Variance reduction through importance sampling:

$$\hat{I} = \frac{1}{N}\sum_{i=1}^{N} f(\mathbf{x}_i)\,, \quad \delta\hat{I} = \sqrt{\frac{1}{N-1}\left(\frac{1}{N}\sum_{i=1}^{N} f^2(\mathbf{x}_i) - \hat{I}^2\right)}$$

$$I = \int_{\mathbf{H}(\mathbf{x})\in\Omega} \mathrm{d}\mathbf{H}\,\frac{f(\mathbf{x})}{h(\mathbf{x})} \qquad h(\mathbf{x}) = \left|\det\left(\frac{\partial \mathbf{H}(\mathbf{x})}{\partial \mathbf{x}}\right)\right|$$

Task: find $h(\mathbf{x})$adaptive MC techniques: VEGAS [Lepage'78], Parni [Hameren'14]

Neural importance sampling

Normalising Flows in Sherpa [Bothmann'20], Madgraph [Heimel'22'25],STRIPPER [Jansen'25]

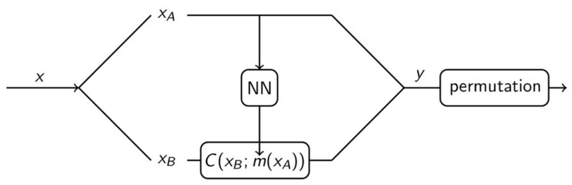
Surrogates

If f(x) is very expensive → finding a surrogate improves performance
→ can be unbiased by unweighting against real f(x)

- QCD factorization aware NN ME emulators [Maitre'21] [Janssen'23]
- two-stage unweighting to correct surrogates [Danziger'22]
- one-loop emulators [Aylett-Bullock'21][Badger'23] [Maitre'23]

Unweighting with Normalizing Flows

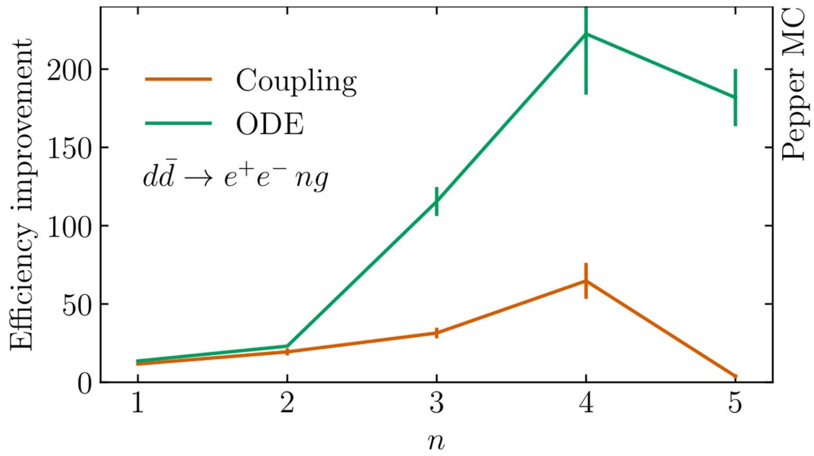
Coupling Layer Flows → series of simple bijections

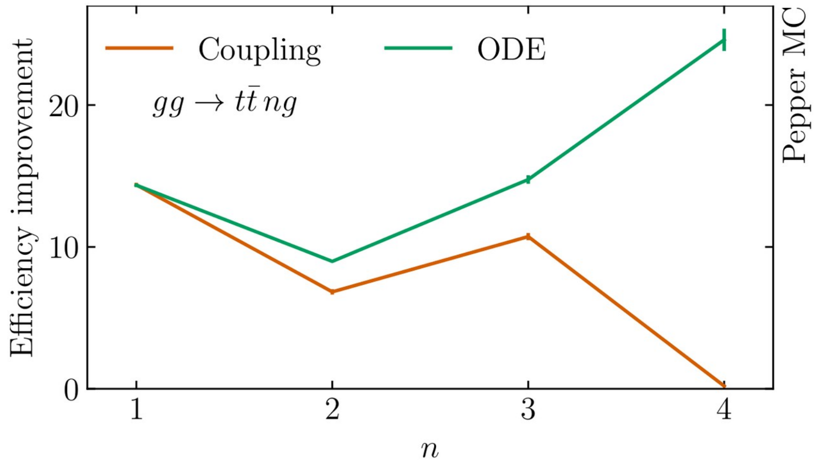


Continuos Normalising Flows $\frac{\mathrm{d}\psi_t(x)}{\mathrm{d}t} = v_t(\psi_t(x))$

Substantial unweighting efficiency improvements

$$\epsilon_\theta = \frac{\langle w_\theta \rangle}{w_{\max,\theta}} \quad \text{with} \quad w_\theta(x) := \frac{p_0(x)}{q_\theta(x)} .$$



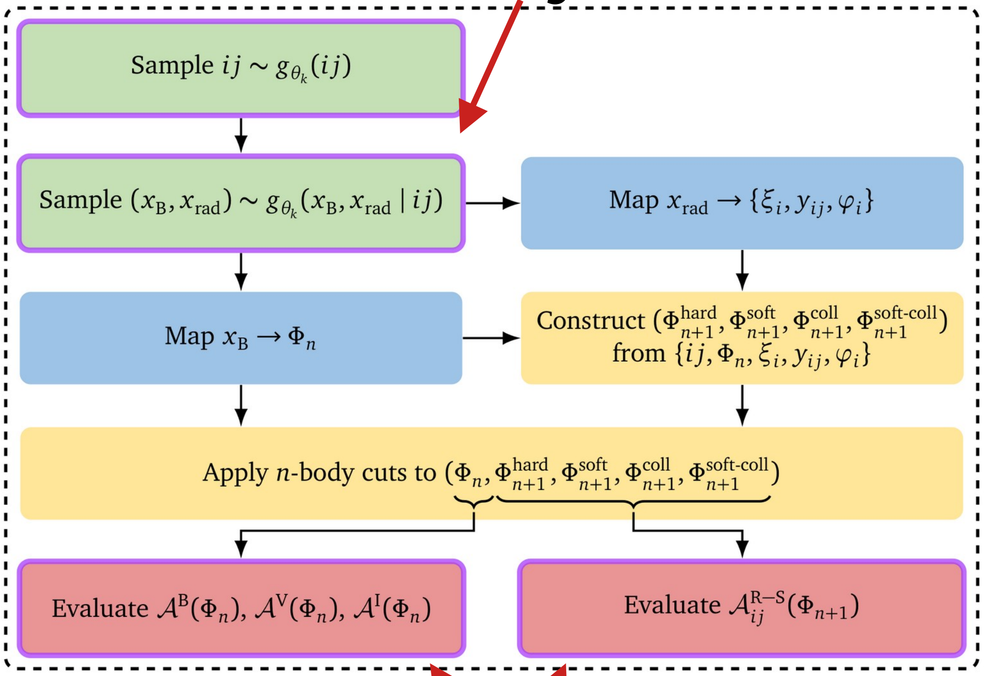


[Bothmann, Janßen, Knobbe, Schmitzer, Sinz, 2604.03511]

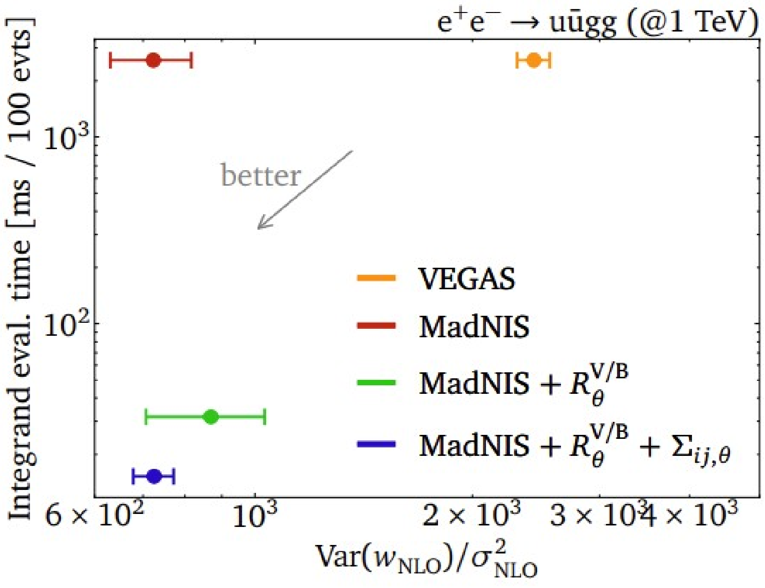
Surrogates + Normalising flows

Madnis@NLO [De Crescenzo, Mariño, Elmer, Heimel, Plehn, Winterhalder, Zaro 2603.22407]

Normalizing Flows



Surrogates

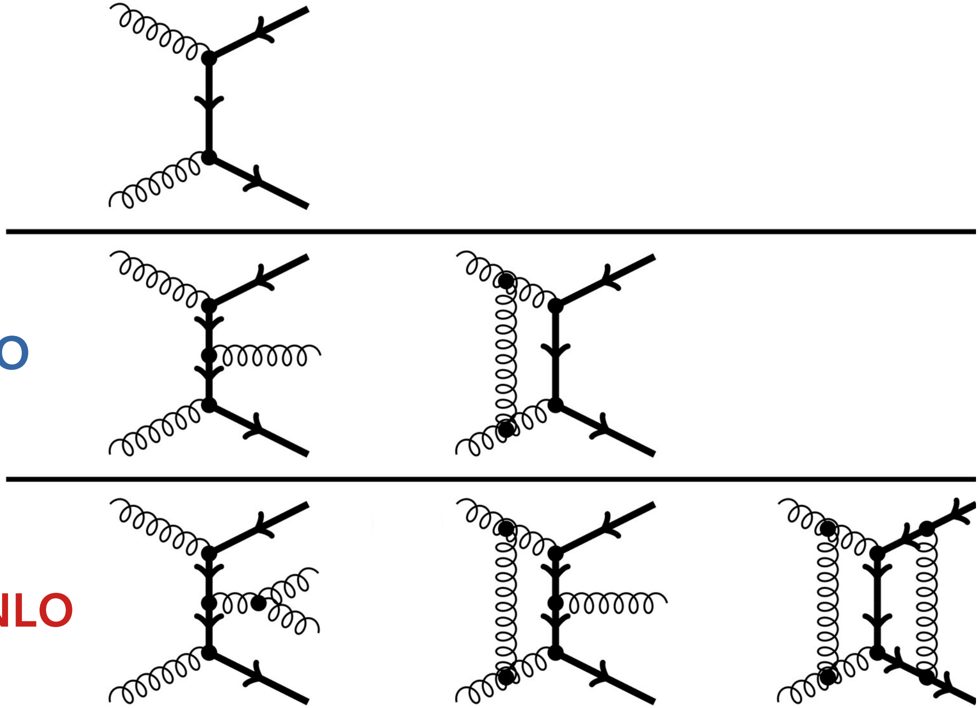


NNLO cross section contributions

LO

NLO

NNLO



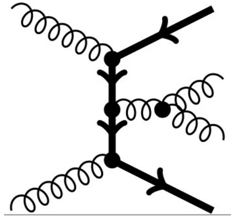
IR-finite cross section

NNLO subtraction schemes

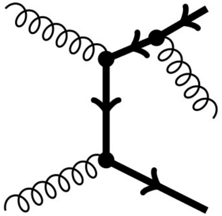
qT-slicing [Catani'07]
N-jettiness slicing [Gaunt'15/Boughezal'15]
Antenna [Gehrmann'05-'08],
Colorful [DelDuca'05-'15]
Sector-improved residue subtraction [Czakon'10-'14,'19]
Projection-to-Born [Cacciari'15]
Nested collinear [Caola'17]
Local Analytic [Magnea'18]
Geometric [Herzog'18]
Unsubtraction [Aguilera-Verdugo'19]

Anatomy of a fixed-order real emission event

Matrix elements | Kinematics + weight | Observable

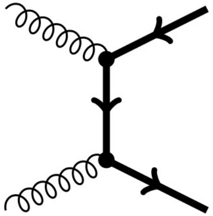


$$\Phi_{n+2},\ w_{n+2} \longrightarrow \mathcal{O}(\Phi_{n+2})$$

$-$  $\otimes \mathcal{S}_1$

$$\Phi_{n+1},\ w_{n+1} \longrightarrow \mathcal{O}(\Phi_{n+1})$$

generic soft/splitting functions

$-$  $\otimes \mathcal{S}_2$

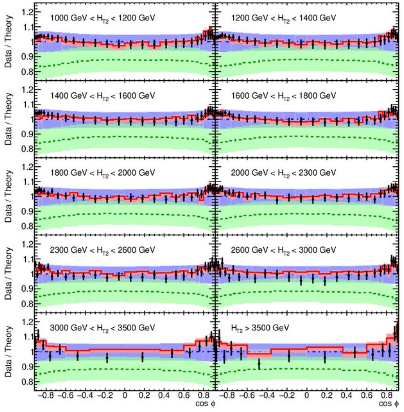
$$\Phi_{n},\ w_{n} \longrightarrow \mathcal{O}(\Phi_{n})$$

$$\boxed{w_{n+2} + w_{n+1} + w_n = \text{finite}}$$

In all infrared limits

Computational costs of fixed-order cross sections

- Normally evaluated as **weighted** MC integrals
  (optimisations for efficient integration like VEGAS implied)
  → histogram cross sections
  (if you change bin edges you start from scratch)
- Can be computationally very challenging
- Negative subtraction source of large **variance/MC error**
  [ATLAS 2301.09351]



→ O(100 MCPUh)

Costs for typical NNLO calc:

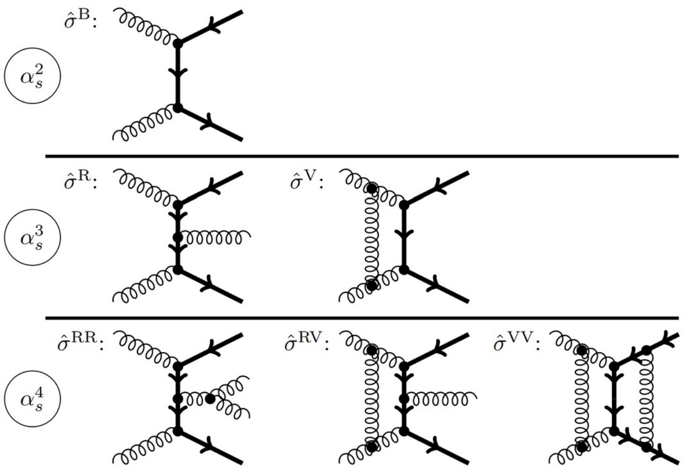
Process class	Core-hours
$pp \to V$	~hours
$pp \to VV$	O(1k)
$pp \to V + j$	O(>10k)
$pp \to jj$	O(>100k)
$pp \to jjj$	O(>1M)

How to make this more efficient?
→ can ML help?

Application to NNLO QCD cross sections

Benchmarks

- To make life simple: optimization goal is total cross section
  → focusing on **gluonic top-pair production**
  (small number of channels)



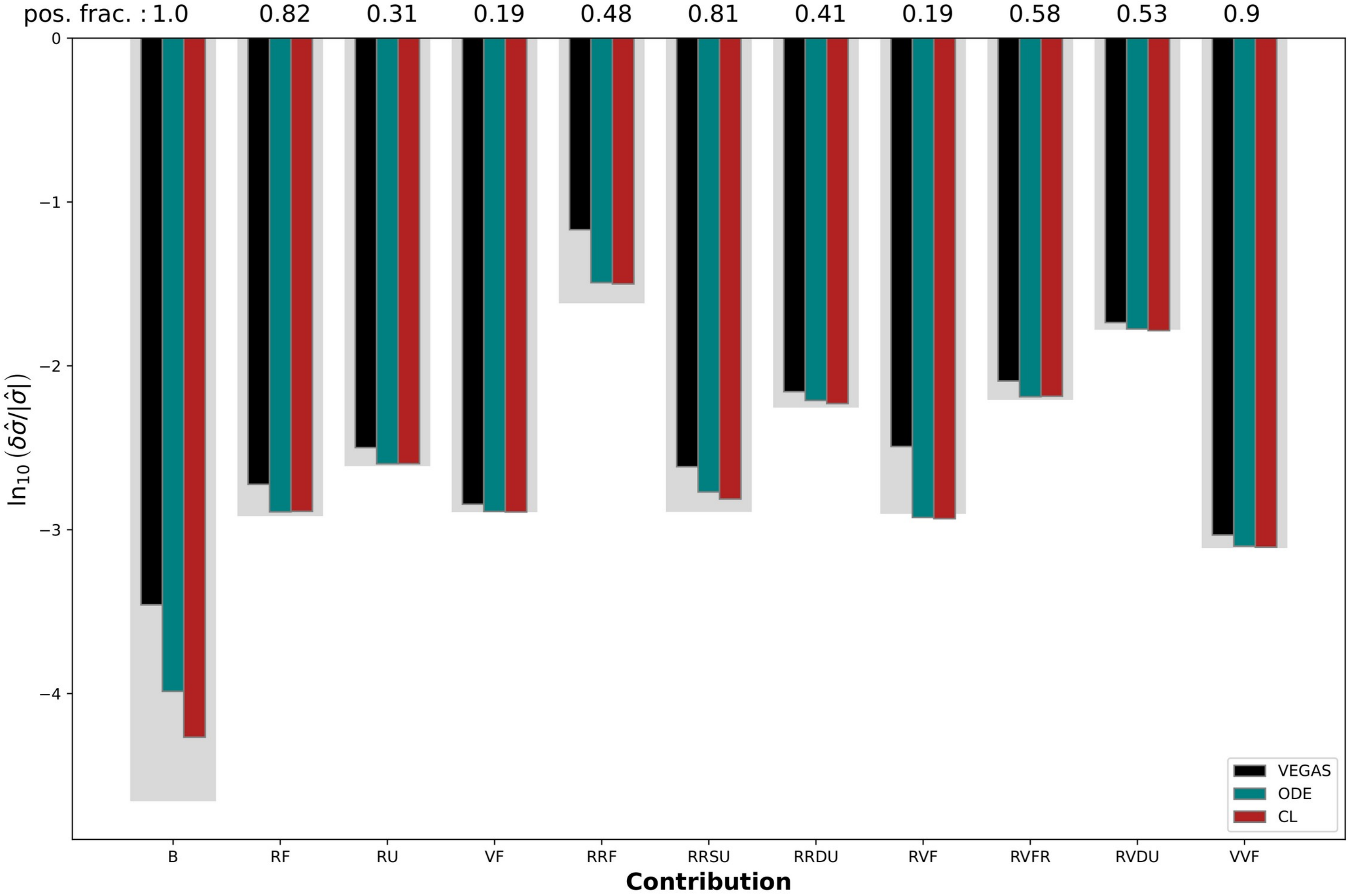
- Three integrators:
  VEGAS (reference for the current default), **CL, ODE**
  → different implementation of discrete parameters:
  continuous and embedded (flows only)

- Benchmark quantities:
  - Weight variance / MC error $$\delta\hat{I} = \sqrt{\frac{1}{N-1}\left(\frac{1}{N}\sum_{i=1}^{N} f^2(\mathbf{x}_i) - \hat{I}^2\right)}$$
  - Unweighting efficiency $$\approx \frac{\langle w\rangle}{w_{\max}}$$

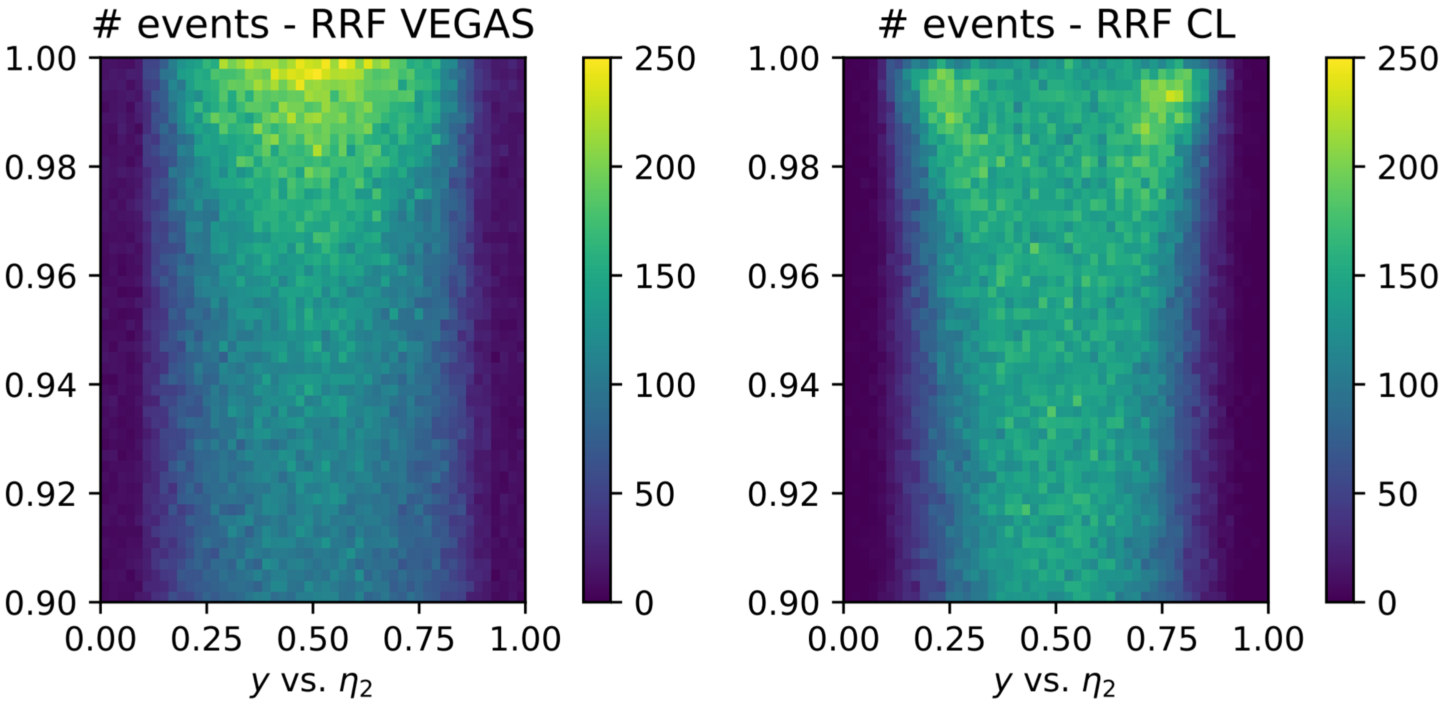
NNLO cross section results

Frozen integrators
1M points

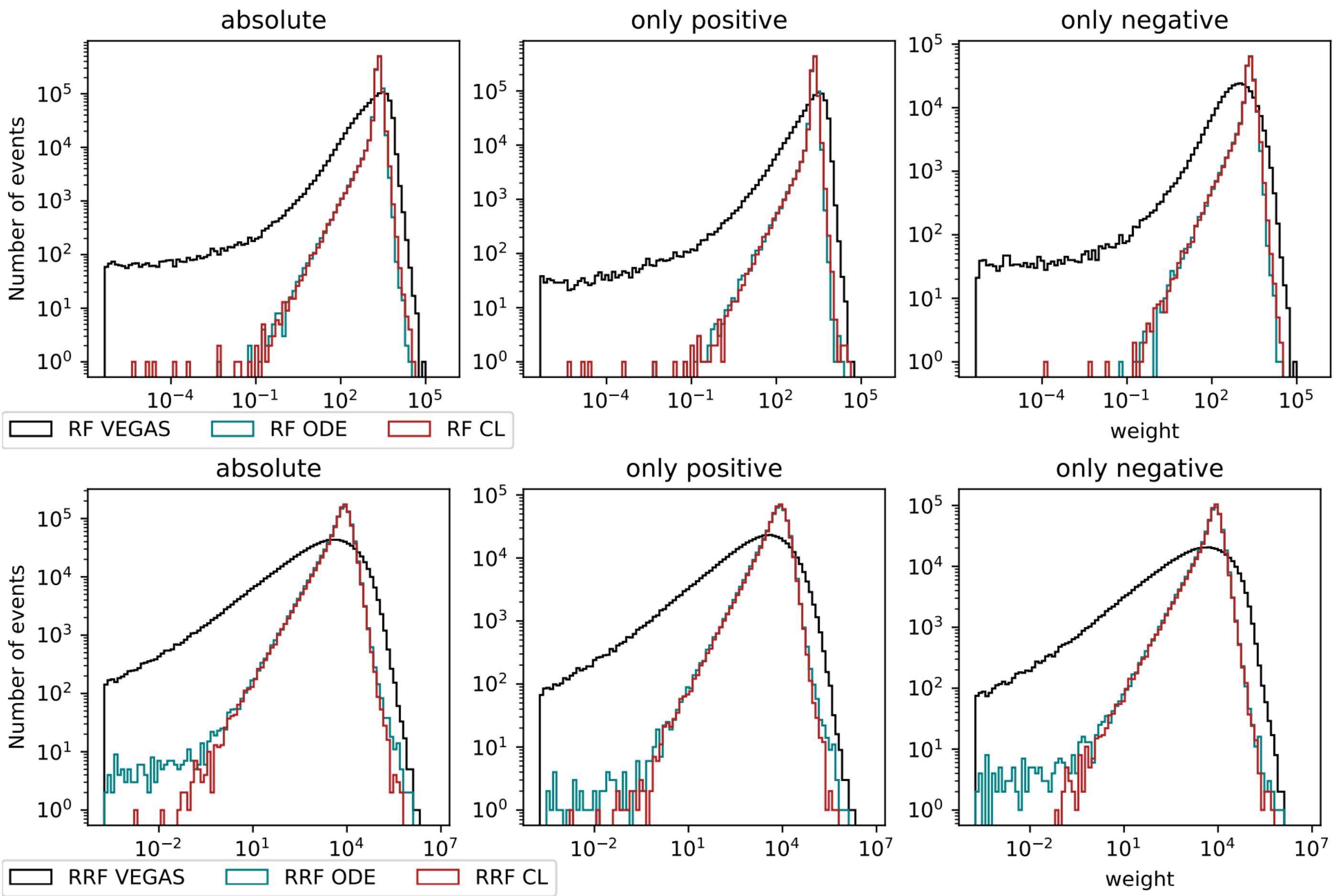
Normalizing Flows appear not to be a game changer**⁉**



Non-factorizing phase space features



Weight distributions



This is a **log-plot**

→ suggests that the variance with NFs are **much better**

Non-positive definite integrands

- Non-definite integrands introduce new challenges

  → cancellation between +/- parts increase the variance

- Consider extreme case: $|f(x)|/h(x) = w = \text{const.}$

  MC estimate:

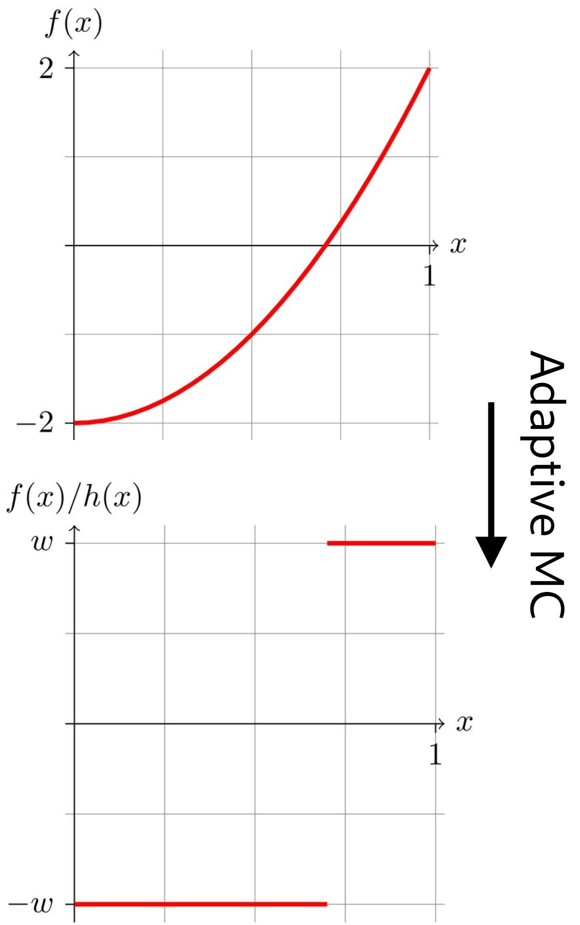
$$\hat{I} = w\frac{N_+ - N_-}{N} \equiv w(2\alpha - 1) \qquad \alpha = N_+/N$$

- Lower bound on variance:

$$\boxed{\mathrm{Var}(\hat{I}) = w^2 - w^2(2\alpha-1)^2 = w^2(4\alpha(1-\alpha))}$$

→ relative uncertainty: $\frac{\delta\hat{I}}{\hat{I}} = \frac{1}{\sqrt{N-1}}\frac{\sqrt{\alpha(1-\alpha)}}{\alpha - \frac{1}{2}}$

Rephrased: at some point it doesn't matter any more how good your adaptive MC is...

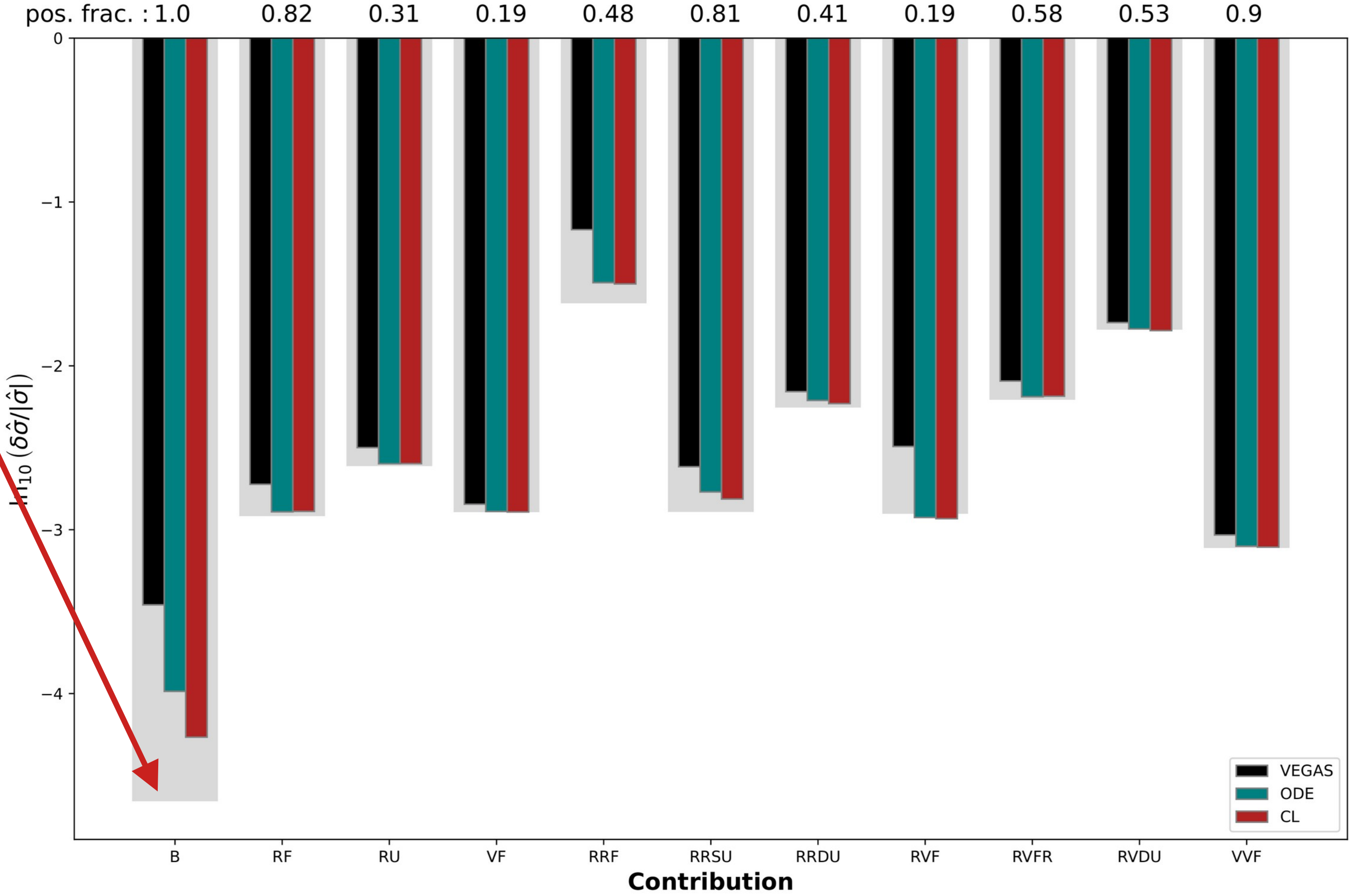


NNLO cross section results → ‘optimal’ uncertainty

Grey boxes
‘Optimal’ uncertainty!

Observation #1:
Saturation of lower bounds both with NF & VEGAS
(for NNLO → NF slightly better)

side remark/interpretation:
1) compliment PS parametrization and/or
2) too simple a problem



Stratification of signed integrands

There are ways around:

1) Add a large constant

2) Stratification: $f(\mathbf{x}) = f_+(\mathbf{x}) + f_-(\mathbf{x})\,, \quad \text{with} \quad f_\pm(\mathbf{x}) = \Theta(\pm f(\mathbf{x}))f(\mathbf{x})$

$$I = \int_{\mathbf{H}_+(\mathbf{x})\in\Omega} \mathrm{d}\mathbf{H}_+ \frac{f_+(\mathbf{x})}{h_+(\mathbf{x})} + \int_{\mathbf{H}_-(\mathbf{x})\in\Omega} \mathrm{d}\mathbf{H}_- \frac{f_-(\mathbf{x})}{h_-(\mathbf{x})}$$

"two independent integrals"

$$\hat{I}_{\text{strat}} = \hat{I}_+ + \hat{I}_- = \frac{1}{N_+}\sum_{i=1}^{N_+}\frac{f_+(\mathbf{x}_i)}{h_+(\mathbf{x}_i)} + \frac{1}{N_-}\sum_{i=1}^{N_-}\frac{f_-(\mathbf{x}_i)}{h_-(\mathbf{x}_i)}$$

$$\delta\hat{I}_{\text{strat}} = \sqrt{\frac{1}{N-1}\left[\frac{N}{N_+}\mathrm{Var}(\hat{I}_+) + \frac{N}{N_-}\mathrm{Var}(\hat{I}_-)\right]}$$

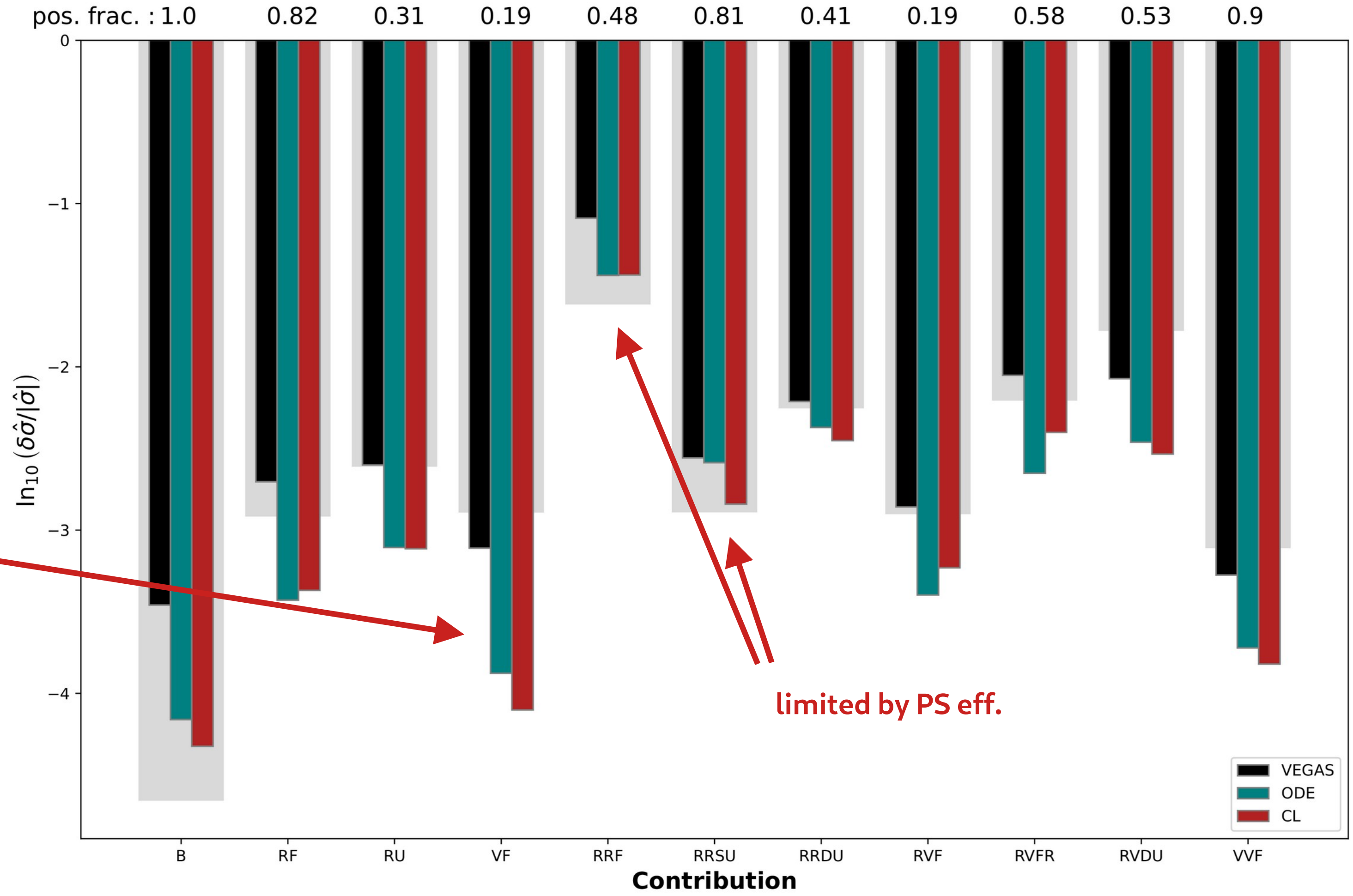
$$\mathrm{Var}(\hat{I}_\pm) = \frac{1}{N_\pm}\sum_{i=1}^{N_\pm}\left(\frac{f_\pm(\mathbf{x}_i)}{h_\pm(\mathbf{x}_i)}\right)^2 - \hat{I}_\pm^2$$

+ The total variance is now bounded by the individual variances

- The mappings are more complicated (need high phase space efficiency)

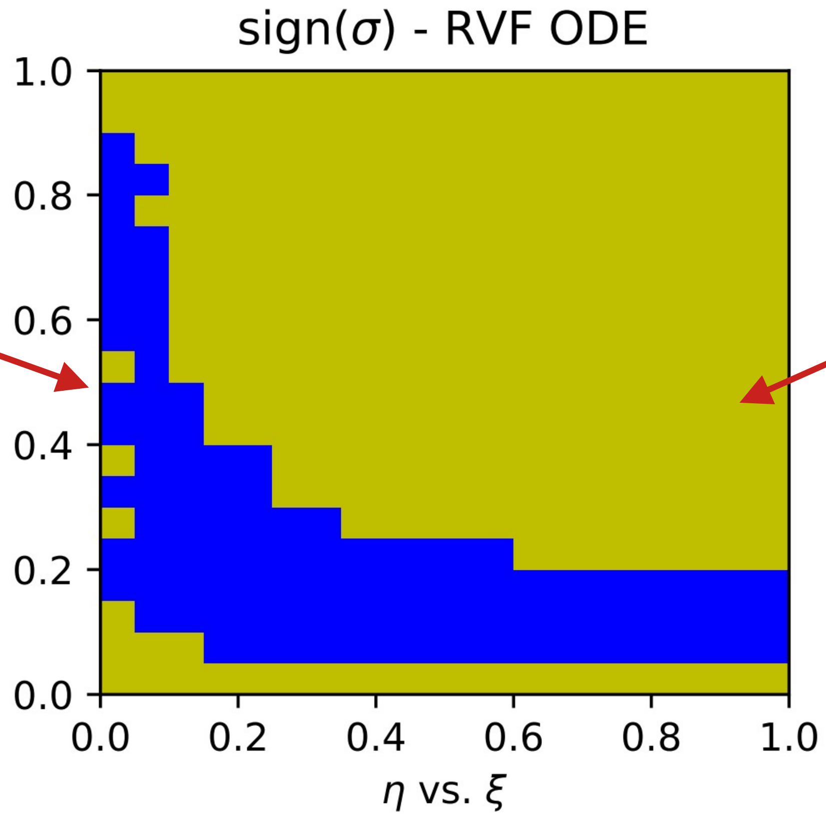
Relative uncertainties with positive/negative split

Observation #2:
Splitting can give a significant performance gains for flow based integrators



Non-trivial positive/negative structure

Negative



Positive

non-factorizing
→ VEGAS not sufficient

Folding in GPU/CPU costs

Final result is sum over contributions:

$$\delta\sigma_{\text{target}} = \delta\hat{\sigma}_{\text{tot}}(\mathbf{n}) = \sqrt{\sum_i (\delta\hat{\sigma}^i(n_i))^2} = \sqrt{\sum_i \frac{\text{Var}(\hat{\sigma}^i)}{n_i - 1}}$$

Minimize the total computational cost: $C = \sum_i C_i n_i$

(→ dominated in the end by the double real contributions)

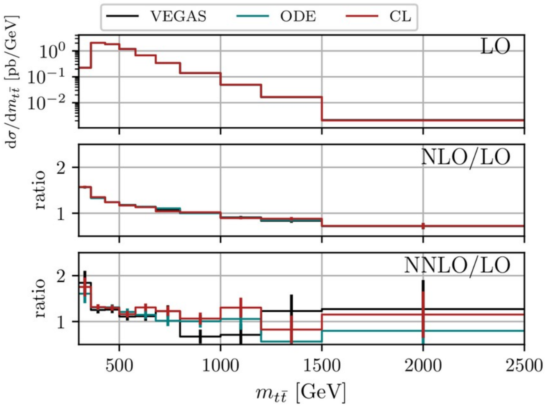
Optimal performance gains (infinity cheap NF/expensive integrand):

- **Absolute integrand training: ~2**
- **Pos/neg split: ~8**

[In our setup: 1-1.5h GPU vs. 10h CPU]

↓	↓
Flow dominated by disk I/O	Integrand dominated by ME

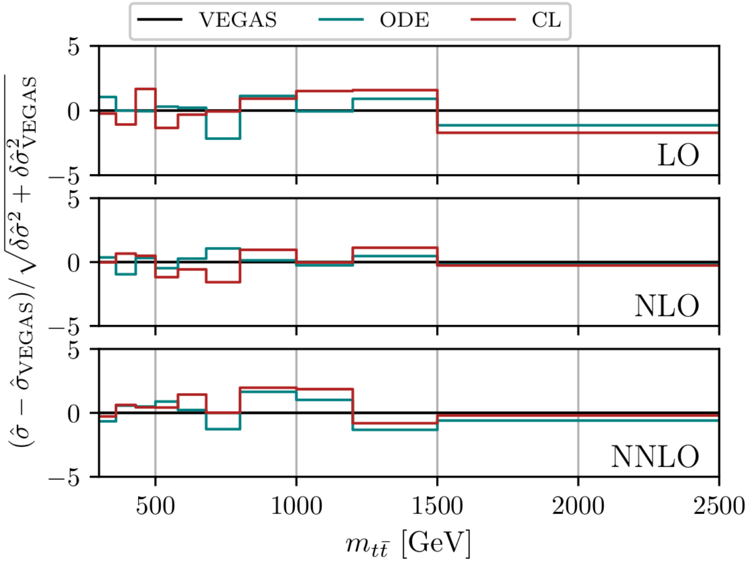
Differential distributions

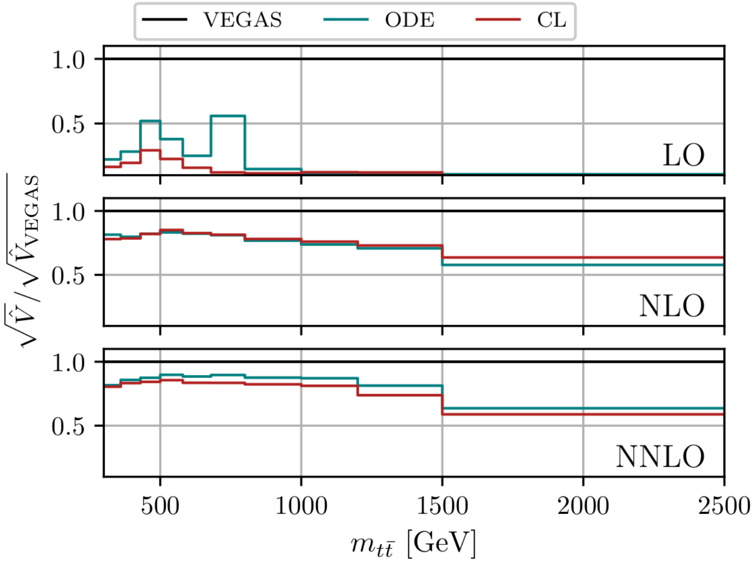


← Steeply falling spectrum

Validation
→ method works differentially

Performance gain limited
by non-definiteness





Summary

- Normalizing flows as a tool for variance reduction
- Investigated the usage of flow-based methods for NNLO QCD phase space sampling
  → Reduction of weight distributions
  → Pos/Neg cancellation bottleneck in many cases → lower bound as for unweighted samples

- Pos/Neg splitting allows **to go beyond this bound**

- Performance gains of O(10) with pos/neg splitting + normalising flow can be achieved
  → **non-factorising model is crucial**!

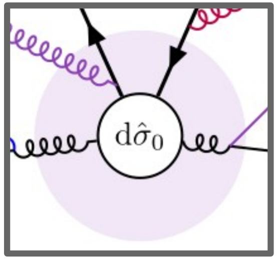
Outlook (wip):

→ fully fledged integration in STRIPPER (& HighTEA)

→ go beyond "simple" NNLO QCD top-pair

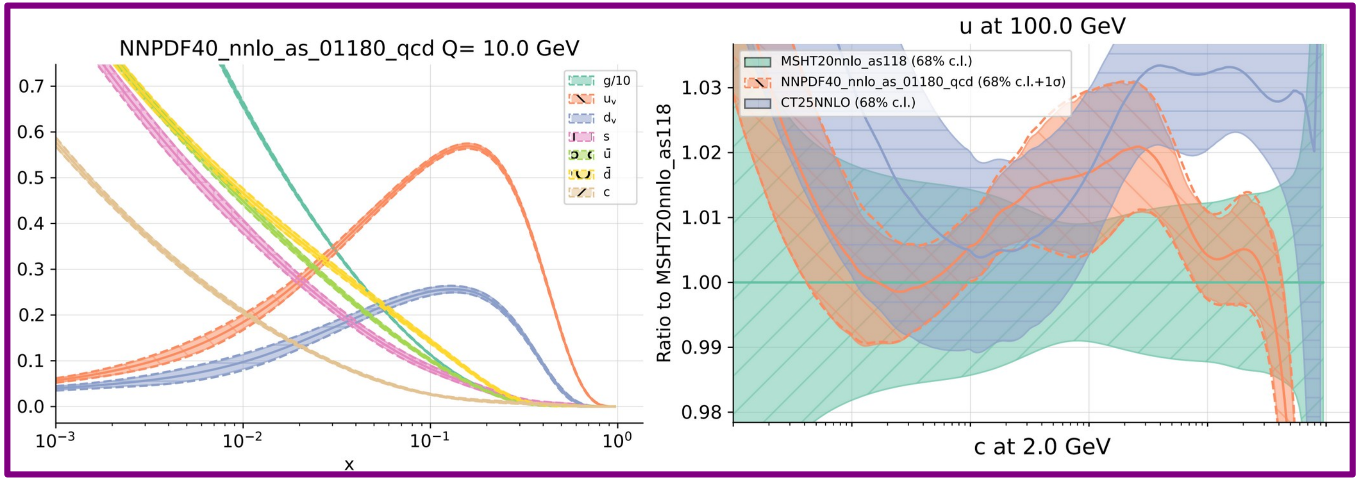
→ differential pos/neg splitting

Fixed-order QCD in collinear factorization

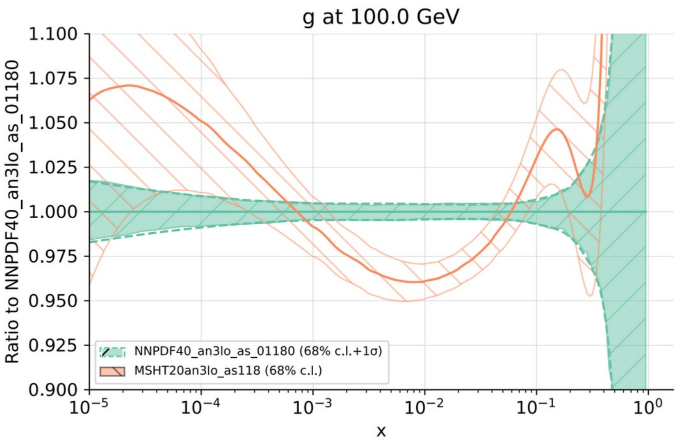


$$\sigma_{h_1 h_2 \to X} = \sum_{ij} \int_0^1 \int_0^1 \underline{\mathrm{d}x_1 \mathrm{d}x_2 \phi_{i,h_1}}(x_1, \mu_F^2)\phi_{j,h_2}(x_2, \mu_F^2)\hat{\sigma}_{ij\to X}(\alpha_s(\mu_R^2), \mu_R^2, \mu_F^2)$$

Collinear (unpolarised) parton distribution functions



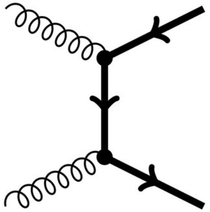
First steps towards N3LO ...

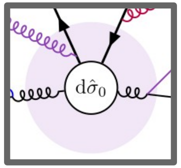


[from Juan's DIS talk last week]

Fixed-order QCD in collinear factorization

LO



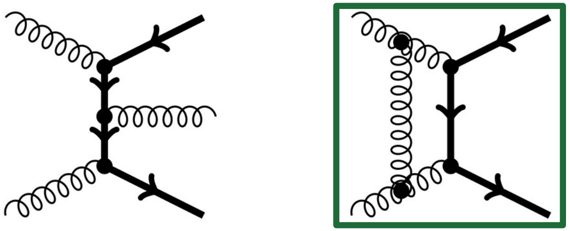


$$\sigma_{h_1h_2\to X} = \sum_{ij}\int_0^1\int_0^1 \mathrm{d}x_1\mathrm{d}x_2\phi_{i,h_1}(x_1,\mu_F^2)\phi_{j,h_2}(x_2,\mu_F^2)\hat{\sigma}_{ij\to X}(\alpha_s(\mu_R^2),\mu_R^2,\mu_F^2)$$

partonic cross section

$$\hat{\sigma}_{ab\to X} = \hat{\sigma}^{(0)}_{ab\to X} + \hat{\sigma}^{(1)}_{ab\to X} + \hat{\sigma}^{(2)}_{ab\to X} + \mathcal{O}(\alpha_s^3)$$

NLO



$$\hat{\sigma}^{(i)} \ni \frac{1}{\mathcal{F}}\int |\mathcal{M}_n|^2 \mathrm{d}\Phi_n$$

IR-finite cross section

NNLO

Automated fixed-order @ NLO

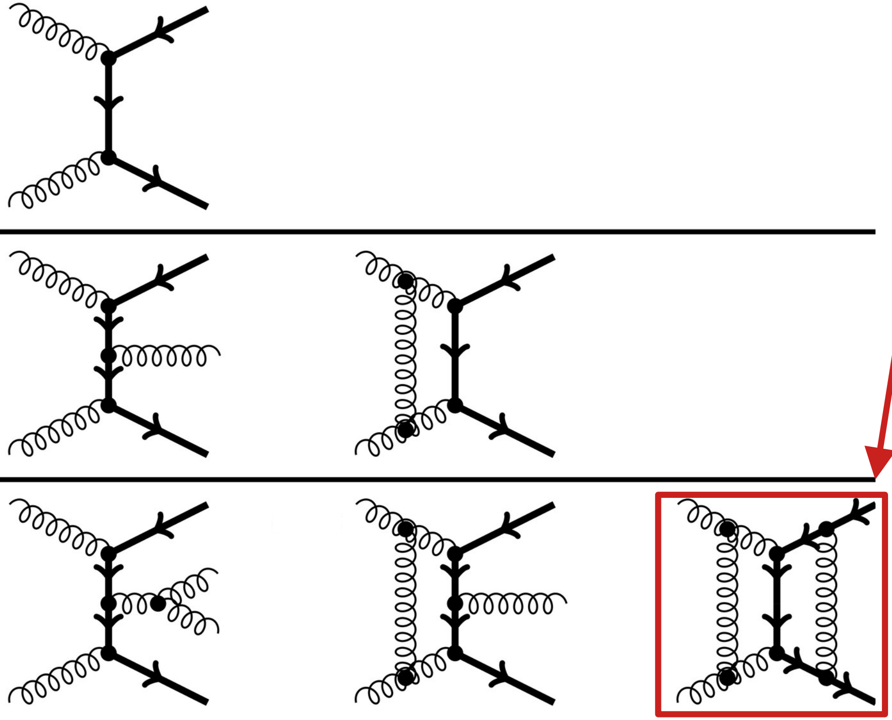
- oneloop-matrix elements
  - Openloops, Recola, Helac, MadGraph, GoSam
- local QCD/EW subtraction
  - FKS: MadGraph5_aMC@NLO, Powheg
  - CS dipoles: Sherpa, Herwig

NNLO QCD challenges: two-loop amplitudes

LO

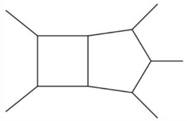
NLO

NNLO

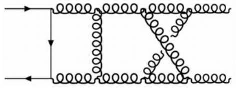


How to compute
multi-scale two-loop QCD amplitudes?
→ **fast growing complexity:**
rational coef. and **special functions**
→ deeper understanding of the analytical properties
→ refinement of computational tools

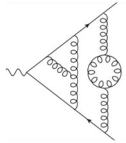
Two-loop 5-point
[Abreu, Agarwal, Badger, Buccioni, Chawdhry, Chicherin,Czakon, de Laurenties, Febres-Cordero, Gambuti, Gehrmann, Henn, Lo Presti, Manteuffel, Ma, Mitov, Page, Peraro, Poncelet, Schabiner Sotnikov, Tancredi, Zhang,...]



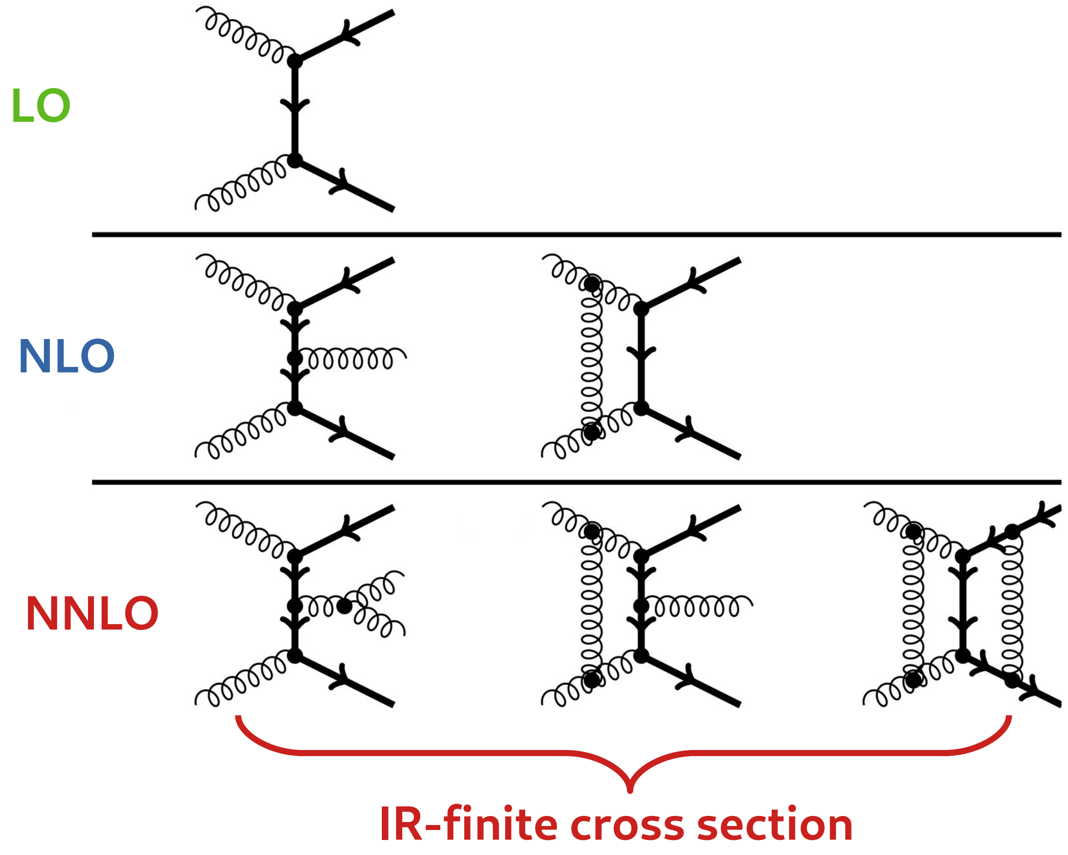
Three-loop 4-point
[Bargiela, Dobadilla, Canko, Caola, Jakubcik, Gambuti, Gehrmann, Henn, Lim, Mella, Mistleberger, Wasser, Manteuffel, Syrrakow, Smirnov, Trancredi, ...]



Four-loop 3-point
[Henn, Lee, Manteuffel, Schabinger, Smirnov, Steinhauser,...]



NNLO QCD challenges: subtraction



How to achieve **infrared (IR) finite differential** cross sections at NNLO QCD?

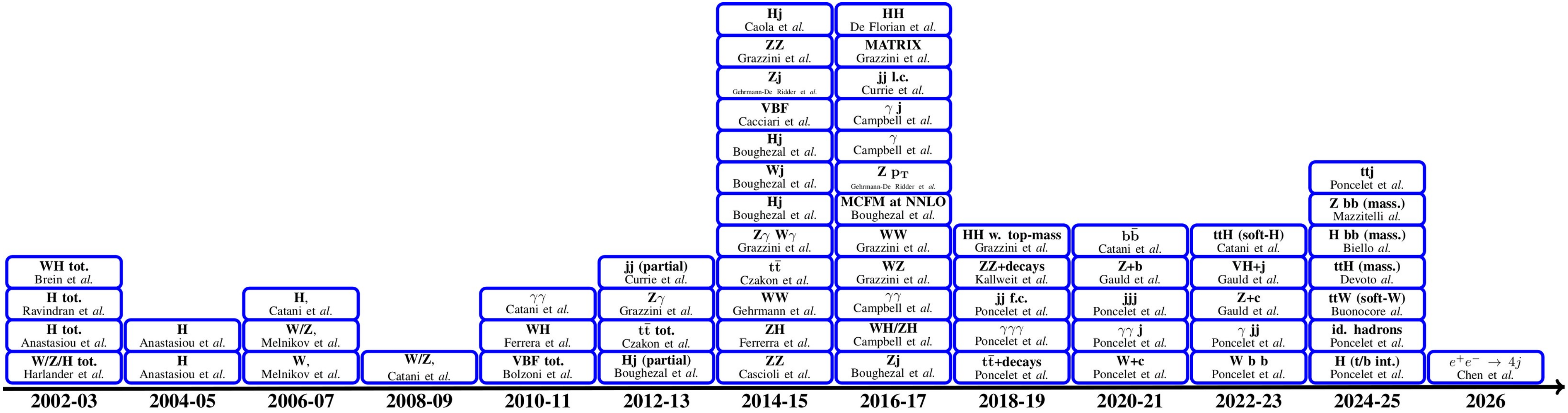
~20 years to solve this problem

qT-slicing [Catani'07], N-jettiness slicing [Gaunt'15/Boughezal'15], Antenna [Gehrmann'05-'08], Colorful [DelDuca'05-'15], Projetction [Cacciari'15], Geometric [Herzog'18], Unsubtraction [Aguilera-Verdugo'19], Nested collinear [Caola'17], Local Analytic [Magnea'18] Sector-improved residue subtraction [Czakon'10-'14,'19]

Automation of NNLO QCD subtraction:

- qT-slicing (massive/color singlet) in **MATRIX** & **MCFM** (public)
- Antenna subtraction in **NNLOJet** (public)
- Sector-improved residue subtraction in **Stripper** (private impl.)
- color singlets with nested-collinear in **history** (public)

Status of NNLO QCD cross section computations



All 2 → 2 SM processes, 2 → 3+ only **limited by**

→ the available two-loop matrix elements

→ **numerical/computational challenges**

Training and model parameters

Training target based on Kullback–Leibler (KL) divergence:

$$D_{\mathrm{KL}}(p \parallel q_\theta) = \sum_{i=1}^{N} p(\mathbf{y}_i) \log\left(\frac{p(\mathbf{y}_i)}{q_\theta(\mathbf{y}_i)}\right)$$

1) Generate sample & evaluate target function
   (1M points)

2) Do NN optimization step with gradient descend (ADAM)

3) Repeat 1)
   (10 times)

Size of neural networks depends on the dimension of the problem.

→ we investigated 4 to 13 dimensional problems + discrete parameters (conditional NNs):

~ 100k – 10M parameters for CL flows

~ 1M parameter for ODE flows

The integrands

Matrix elements + measurement functions

Structure of partonic contributions:

different partonic channels $\hat{\sigma}^i_{ab} = \sum_{j \in \mathcal{C}^i_{ab}(ab \to X)} \hat{\sigma}^i_{ab,j}$ $\Rightarrow$ $\hat{\sigma}^i_{ab,j} = N_j \int \mathrm{d}\Phi_j \mathcal{F}_{ab,j}(\Phi_j)$

Subtraction for real-emission contributions
based on sector-decomposition + residue subtraction:

$$\hat{\sigma}^{(1)} \ni \int \mathrm{d}\Phi_j \sum_{kl} \mathcal{S}_{kl} |\mathcal{M}_j(\Phi_j)|^2 F(\Phi_j)\,. \qquad \hat{\sigma}^{(i)} \ni \int_{[0,1]^m} \mathrm{d}^m\chi \int_{[0,1]^n} \mathrm{d}^n\mathbf{x} \frac{f^j_{\{k\}}(\chi,\mathbf{x}) - \sum f^j_{\{k\}}(\chi|_{\to 0},\mathbf{x})}{\prod_i^m \chi_i}$$

But for the integration problem (in case of total cross sections) it is enough to consider:

$$\boxed{\hat{\sigma}^{(i)} \ni \int_{[0,1]^n} \mathrm{d}\mathbf{y}\, g^j_{\{k\}}(\mathbf{y})}$$

$$\sum_{\{k\},h} \int_{[0,1]^n} \mathrm{d}\mathbf{y}\, g^j_{\{k\},h}(\mathbf{y}) = \sum_l \int_{[0,1]^n} \mathrm{d}\mathbf{y}\, g^j_l(\mathbf{y})$$

Sum of
Sectors, helicities, channels

Coupling Layer Normalizing Flow

Based on the i-flow paper: 2001.05486

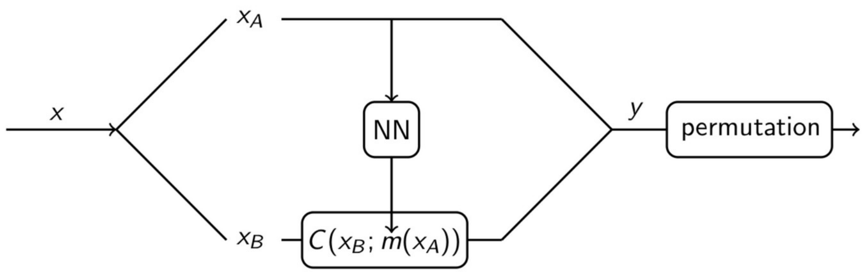
Series of bijections:

$$\vec{x}_K = c_K(c_{K-1}(\cdots c_2(c_1(\vec{x}))))$$

Distribution:

$$g_K(\vec{x}_K) = g_0(\vec{x}_0) \prod_{k=1}^{K} \left| \frac{\partial c_k(\vec{x}_{k-1})}{\partial \vec{x}_{k-1}} \right|^{-1}, \quad \text{where} \quad \begin{cases} \vec{x}_0 = \vec{x} \\ \vec{x}_k = c_k(\vec{x}_{k-1}) \end{cases}$$

Structure of a single coupling layer:



$$\begin{aligned} x'_A &= x_A, & A &\in [1, d]\,, \\ x'_B &= C(x_B; m(\vec{x}_A)), & B &\in [d+1, D]\,. \end{aligned}$$

The inverse map

$$x_A = x'_A \, ,$$
$$x_B = C^{-1}(x'_B; m(\vec{x}'_A)) = C^{-1}(x'_B; m(\vec{x}_A))$$

Inverse Jacobian: $\left|\frac{\partial c(\vec{x})}{\partial \vec{x}}\right|^{-1} = \left|\begin{pmatrix} \vec{1} & 0 \\ \frac{\partial C}{\partial m}\frac{\partial m}{\partial \vec{x}_A} & \frac{\partial C}{\partial \vec{x}_B} \end{pmatrix}\right|^{-1} = \left|\frac{\partial C(\vec{x}_B; m(\vec{x}_A))}{\partial \vec{x}_B}\right|^{-1}$

Derivative of the NN

Big advantage: $\frac{\partial m}{\partial x_A}$ not needed!
→ Performance

How many coupling layers you need? D for $D \leq 5$ & $2\log_2 D$ for $D > 5$

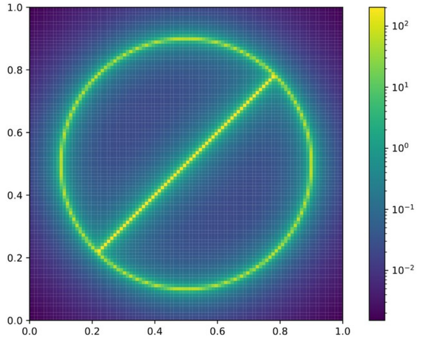
Example: Masking to capture all correlations for D=12

	Dimension	0	1	2	3	4	5	6	7	8	9	10	11
2x	Transformation 1	0	1	0	1	0	1	0	1	0	1	0	1
	Transformation 2	0	0	1	1	0	0	1	1	0	0	1	1
	Transformation 3	0	0	0	0	1	1	1	1	0	0	0	0
	Transformation 4	0	0	0	0	0	0	0	0	1	1	1	1

A toy example

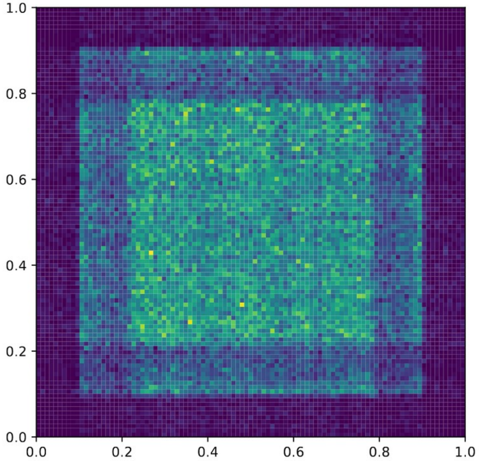
Multi-modular function ("stop-sign"):

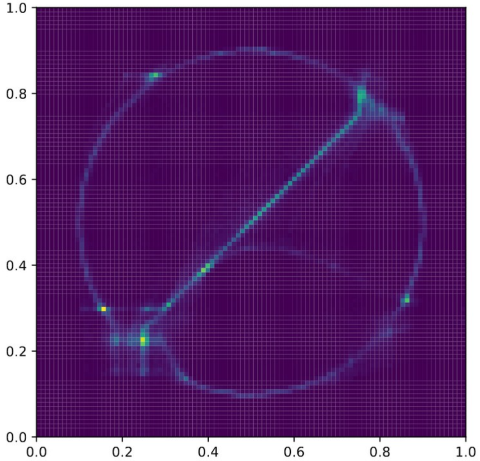
$$f(x,y) = \frac{1}{2\pi^2} \cdot \frac{\Delta r}{\left(\sqrt{(x-x_0)^2+(y-y_0)^2} - r_0\right)^2 + (\Delta r)^2} \cdot \frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2}}$$
$$+\frac{1}{2\pi r_0} \cdot \frac{\Delta r}{((y-y_0)-(x-x_0))^2+(\Delta r)^2} \cdot \Theta\left(r_0 - \sqrt{(x-x_0)^2+(y-y_0)^2}\right).$$



Sampling densities:

VEGAS





Coupling-Layer Flow

Continuous Flows

"time"-dependent probability density function: $\log q_t(\mathbf{y}_t) = \log q_0(\mathbf{y}_0) - \log\left|\det \frac{\partial \phi_t}{\partial \mathbf{y}_0}\right|$

constructed from $\frac{\mathrm{d}}{\mathrm{d}t}\phi_t(\mathbf{y}) = v_t(\phi_t(\mathbf{y}))\,, \quad \text{with} \quad \phi_0(\mathbf{y}) = \mathbf{y}$

+continuity equation (preserve prob.)

vector-field is given by a trainable NN

The mapped point is the solution of a simple ODE: $\mathbf{y}_1 = \phi_1(\mathbf{y}_0) = \int_0^1 v_t(\phi_t(\mathbf{y}_0))\,\mathrm{d}t$

Jacobian by inverse ODE:

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{bmatrix}\phi_{1-s}(\mathbf{y}) \\ f(1-s)\end{bmatrix} = \begin{bmatrix}-v_{1-s}(\phi_{1-s}(\mathbf{y})) \\ -\operatorname{div}\big(v_{1-s}(\phi_{1-s}(\mathbf{y}))\big)\end{bmatrix} \qquad \begin{bmatrix}\phi_1(\mathbf{y}) \\ f(1)\end{bmatrix} = \begin{bmatrix}\mathbf{y}_1 \\ 0\end{bmatrix}$$

$$\rightarrow \quad \log q_1(\mathbf{y_1}) = \log q_0(\mathbf{y_0}) - f(0)$$

Results LO and NLO

Contribution	$\sigma_{\mathrm{B}} \cdot 10^{-2}$	$\sigma_{\mathrm{RF}} \cdot 10^{-2}$	$\sigma_{\mathrm{RU}} \cdot 10^{-2}$	$\sigma_{\mathrm{VF}} \cdot 10^{-1}$
δ^{opt}	0.0001	0.008	0.01	0.006
CPU cost [a.u.]	1	6	1.5	1.3
VEGAS				
$\hat{\sigma} \pm \delta\hat{\sigma}$	5.3278 ± 0.002	6.967 ± 0.01	-5.378 ± 0.02	-4.487 ± 0.006
$\hat{\sigma}^+ \pm \delta\hat{\sigma}^+$	5.3284 ± 0.002	8.955 ± 0.01	4.432 ± 0.006	1.403 ± 0.0008
$\hat{\sigma}^- \pm \delta\hat{\sigma}^-$	$-0.0006446 \pm 5e-07$	-1.981 ± 0.004	-9.8098 ± 0.008	-5.8939 ± 0.003
$\Sigma^{\pm}$	5.3278 ± 0.002	6.973 ± 0.01	-5.378 ± 0.01	-4.4909 ± 0.003
ϵ_{Φ}^{+}	0.99	0.728	0.639	0.808
ϵ_{Φ}^{-}	0.834	0.384	0.852	0.95
$\epsilon^+ (\epsilon^+_{0.1\%})$	0.13 (0.43)	0.048 (0.098)	0.02 (0.048)	0.082 (0.21)
$\epsilon^- (\epsilon^-_{0.1\%})$	0.016 (0.066)	0.013 (0.021)	0.049 (0.17)	0.12 (0.27)
ODE Flow				
$\hat{\sigma} \pm \delta\hat{\sigma}$	5.3279 ± 0.0005	6.98 ± 0.009	-5.394 ± 0.01	-4.483 ± 0.006
$\hat{\sigma}^+ \pm \delta\hat{\sigma}^+$	5.32872 ± 0.0004	8.9494 ± 0.002	4.4185 ± 0.002	1.4018 ± 0.0001
$\hat{\sigma}^- \pm \delta\hat{\sigma}^-$	$-0.00064495 \pm 1e-07$	-1.9844 ± 0.0006	-9.802 ± 0.003	-5.89315 ± 0.0005
$\Sigma^{\pm}$	5.32808 ± 0.0004	6.965 ± 0.003	-5.3835 ± 0.004	-4.4914 ± 0.0006
ϵ_{Φ}^{+}	1.0	0.991	0.992	1.0
ϵ_{Φ}^{-}	0.997	0.99	0.987	0.999
$\epsilon^+ (\epsilon^+_{0.1\%})$	0.33 (0.7)	0.025 (0.3)	0.0059 (0.099)	0.11 (0.56)
$\epsilon^- (\epsilon^-_{0.1\%})$	0.055 (0.36)	0.028 (0.17)	0.02 (0.16)	0.12 (0.73)
Coupling Layer Flow				
$\hat{\sigma} \pm \delta\hat{\sigma}$	5.32795 ± 0.0003	6.972 ± 0.009	-5.39 ± 0.01	-4.492 ± 0.006
$\hat{\sigma}^+ \pm \delta\hat{\sigma}^+$	5.32807 ± 0.0003	8.949 ± 0.002	4.4101 ± 0.002	$1.40155 \pm 9e-05$
$\hat{\sigma}^- \pm \delta\hat{\sigma}^-$	$-0.000644883 \pm 3e-08$	-1.9821 ± 0.0007	-9.8 ± 0.002	-5.89183 ± 0.0003
$\Sigma^{\pm}$	5.32742 ± 0.0003	6.9669 ± 0.003	-5.3899 ± 0.004	-4.49028 ± 0.0004
ϵ_{Φ}^{+}	1.0	0.989	0.988	1.0
ϵ_{Φ}^{-}	1.0	0.99	0.994	1.0
$\epsilon^+ (\epsilon^+_{0.1\%})$	0.53 (0.81)	0.028 (0.24)	0.0082 (0.046)	0.17 (0.63)
$\epsilon^- (\epsilon^-_{0.1\%})$	0.11 (0.79)	0.0074 (0.06)	0.009 (0.19)	0.4 (0.79)

Results NNLO

Contribution	$\sigma_{\mathrm{RRF}} \cdot 10^{-2}$	$\sigma_{\mathrm{RRSU}} \cdot 10^{-3}$	$\sigma_{\mathrm{RRDU}} \cdot 10^{-3}$	$\sigma_{\mathrm{RVF}} \cdot 10^{-2}$	$\sigma_{\mathrm{RVFR}} \cdot 10^{-2}$	$\sigma_{\mathrm{RVDU}} \cdot 10^{-3}$	$\sigma_{\mathrm{VVF}} \cdot 10^{-0}$
δ^{opt}	0.0079	0.0036	0.013	0.0081	0.0018	0.0045	0.011
CPU cost [a.u.]	53	26	8	1542	2.7	2.7	13
VEGAS							
$\hat{\sigma} \pm \delta\hat{\sigma}$	-0.3 ± 0.02	2.835 ± 0.0069	-2.461 ± 0.017	-6.808 ± 0.022	0.2864 ± 0.0023	0.267 ± 0.0049	14.768 ± 0.014
$\hat{\sigma}^+ \pm \delta\hat{\sigma}^+$	3.823 ± 0.013	3.718 ± 0.0056	5.639 ± 0.0065	1.669 ± 0.0026	1.051 ± 0.0014	2.381 ± 0.0011	16.711 ± 0.0067
$\hat{\sigma}^- \pm \delta\hat{\sigma}^-$	-4.135 ± 0.013	-0.8813 ± 0.0022	-8.078 ± 0.0085	-8.4539 ± 0.0068	-0.7635 ± 0.0012	-2.1114 ± 0.0012	-1.955 ± 0.0011
$\Sigma^{\pm}$	-0.312 ± 0.025	2.836 ± 0.0079	-2.439 ± 0.015	-6.785 ± 0.0094	0.2871 ± 0.0025	0.2696 ± 0.0023	14.756 ± 0.0078
ϵ_{Φ}^{+}	0.54	0.732	0.803	0.55	0.593	0.946	0.956
ϵ_{Φ}^{-}	0.513	0.402	0.8	0.871	0.501	0.916	0.816
$\epsilon^{+}(\epsilon^{+}_{0.1\%})$	0.004 (0.0052)	0.0073 (0.017)	0.0018 (0.022)	0.013 (0.024)	0.025 (0.06)	0.029 (0.22)	0.015 (0.1)
$\epsilon^{-}(\epsilon^{-}_{0.1\%})$	0.0041 (0.0053)	0.0035 (0.01)	0.0037 (0.019)	0.014 (0.038)	0.021 (0.05)	0.011 (0.11)	0.024 (0.17)
ODE Flow							
$\hat{\sigma} \pm \delta\hat{\sigma}$	-0.333 ± 0.011	2.823 ± 0.0048	-2.421 ± 0.015	-6.784 ± 0.0081	0.2893 ± 0.0019	0.271 ± 0.0046	14.758 ± 0.012
$\hat{\sigma}^+ \pm \delta\hat{\sigma}^+$	3.812 ± 0.0063	3.7056 ± 0.0028	5.6439 ± 0.0036	1.6764 ± 0.00085	1.0495 ± 0.00038	2.3806 ± 0.00043	16.72 ± 0.0026
$\hat{\sigma}^- \pm \delta\hat{\sigma}^-$	-4.142 ± 0.0057	-0.8848 ± 0.0045	-8.071 ± 0.0068	-8.4578 ± 0.0019	-0.76271 ± 0.00026	-2.1109 ± 0.0005	-1.9554 ± 0.00023
$\Sigma^{\pm}$	-0.33 ± 0.012	2.821 ± 0.0073	-2.427 ± 0.01	-6.7814 ± 0.0027	0.2868 ± 0.00064	0.2697 ± 0.00093	14.765 ± 0.0028
ϵ_{Φ}^{+}	0.855	0.945	0.977	0.988	0.981	0.998	0.999
ϵ_{Φ}^{-}	0.853	0.834	0.982	0.991	0.983	0.998	0.998
$\epsilon^{+}(\epsilon^{+}_{0.1\%})$	0.0013 (0.0027)	0.0028 (0.022)	0.003 (0.023)	0.0029 (0.067)	0.0081 (0.099)	0.03 (0.27)	0.0086 (0.49)
$\epsilon^{-}(\epsilon^{-}_{0.1\%})$	0.0019 (0.0053)	0.00025 (0.00025)	0.0018 (0.0096)	0.051 (0.25)	0.017 (0.095)	0.017 (0.18)	0.17 (0.49)
Coupling Layer Flow							
$\hat{\sigma} \pm \delta\hat{\sigma}$	-0.317 ± 0.01	2.829 ± 0.0044	-2.414 ± 0.014	-6.778 ± 0.0079	0.285 ± 0.0019	0.276 ± 0.0045	14.771 ± 0.012
$\hat{\sigma}^+ \pm \delta\hat{\sigma}^+$	3.814 ± 0.0054	3.7038 ± 0.0025	5.6425 ± 0.0039	1.676 ± 0.0024	1.05 ± 0.00059	2.3799 ± 0.00043	16.718 ± 0.0021
$\hat{\sigma}^- \pm \delta\hat{\sigma}^-$	-4.134 ± 0.0063	-0.8849 ± 0.0015	-8.0756 ± 0.0047	-8.456 ± 0.0026	-0.76256 ± 0.00055	-2.1091 ± 0.00036	-1.95479 ± 0.00011
$\Sigma^{\pm}$	-0.32 ± 0.012	2.819 ± 0.0041	-2.433 ± 0.0086	-6.7803 ± 0.005	0.2874 ± 0.0011	0.2708 ± 0.00079	14.763 ± 0.0022
ϵ_{Φ}^{+}	0.838	0.956	0.985	0.987	0.988	0.998	1.0
ϵ_{Φ}^{-}	0.852	0.836	0.988	0.991	0.985	0.999	1.0
$\epsilon^{+}(\epsilon^{+}_{0.1\%})$	0.0022 (0.0056)	0.0096 (0.022)	0.0029 (0.011)	0.00076 (0.00076)	0.0024 (0.035)	0.01 (0.22)	0.02 (0.31)
$\epsilon^{-}(\epsilon^{-}_{0.1\%})$	0.0034 (0.006)	0.0012 (0.0028)	0.0044 (0.011)	0.0054 (0.13)	0.0017 (0.018)	0.022 (0.22)	0.1 (0.68)