State of the art higher order calculations for LHC physics

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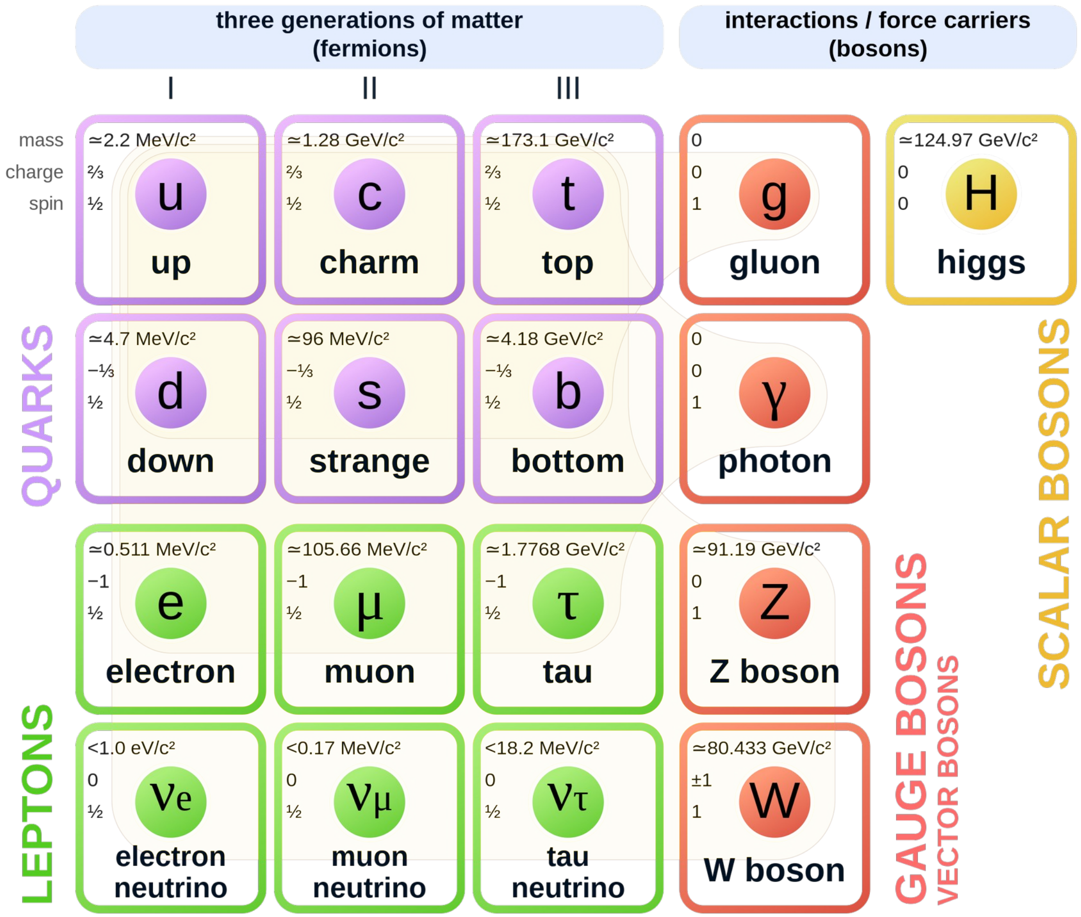
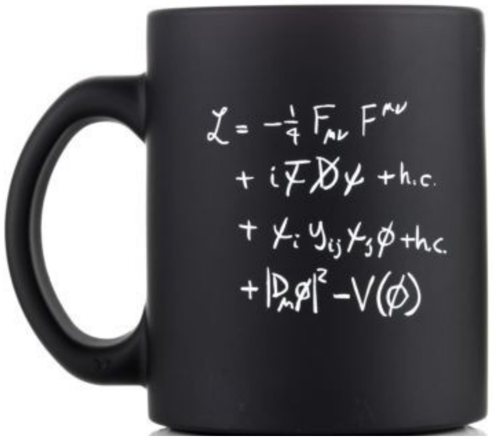
Outline

- Why higher-order perturbation theory?
- The state of the art
  - Where are we? Where are we going?
- Where are the challenges? → techniques & examples
  - Multi-loop amplitudes
  - Subtraction
  - Numerics

Standard Model of Elementary Particles



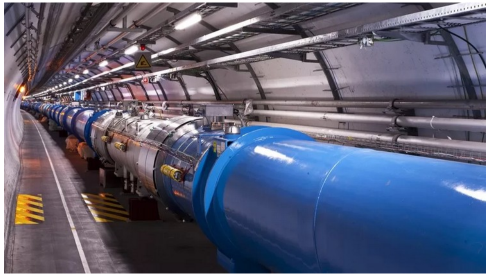


Credit: Wikipedia/CERN

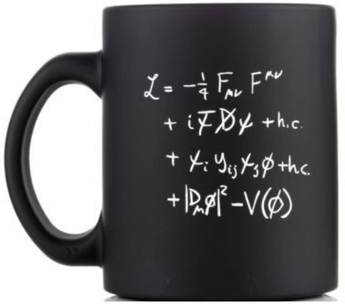
What are the fundamental building blocks of matter?

Scattering experiments

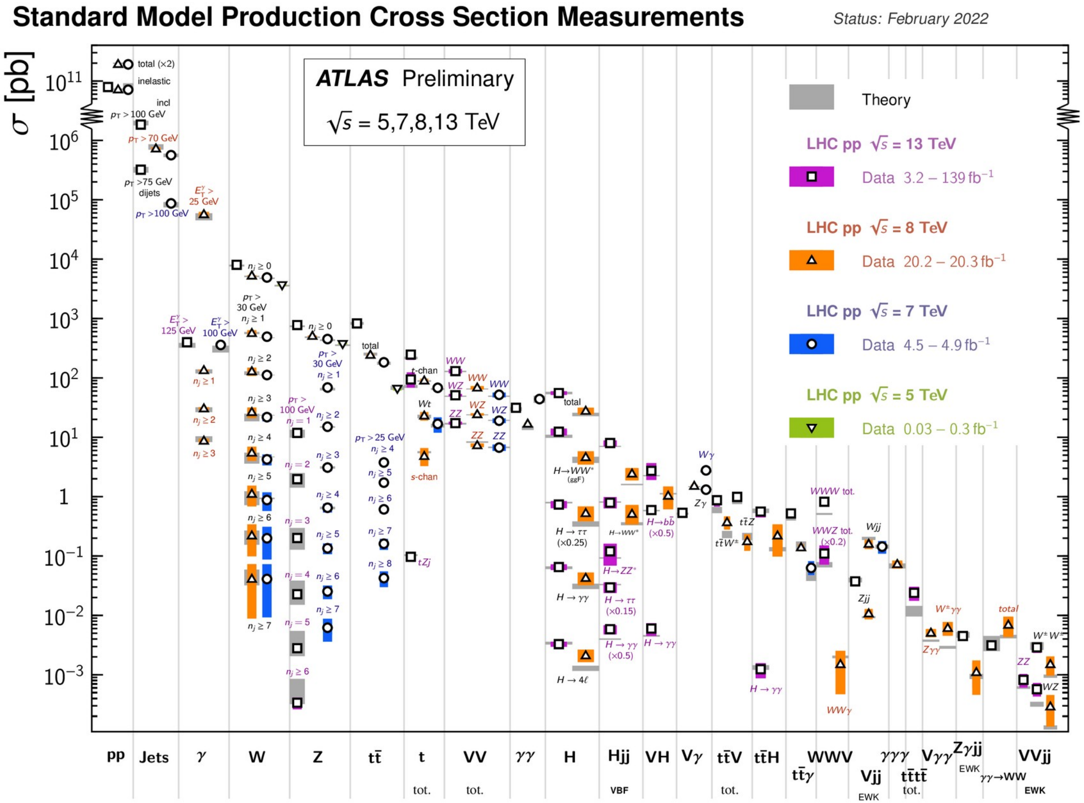
Large Hadron Collider (LHC)



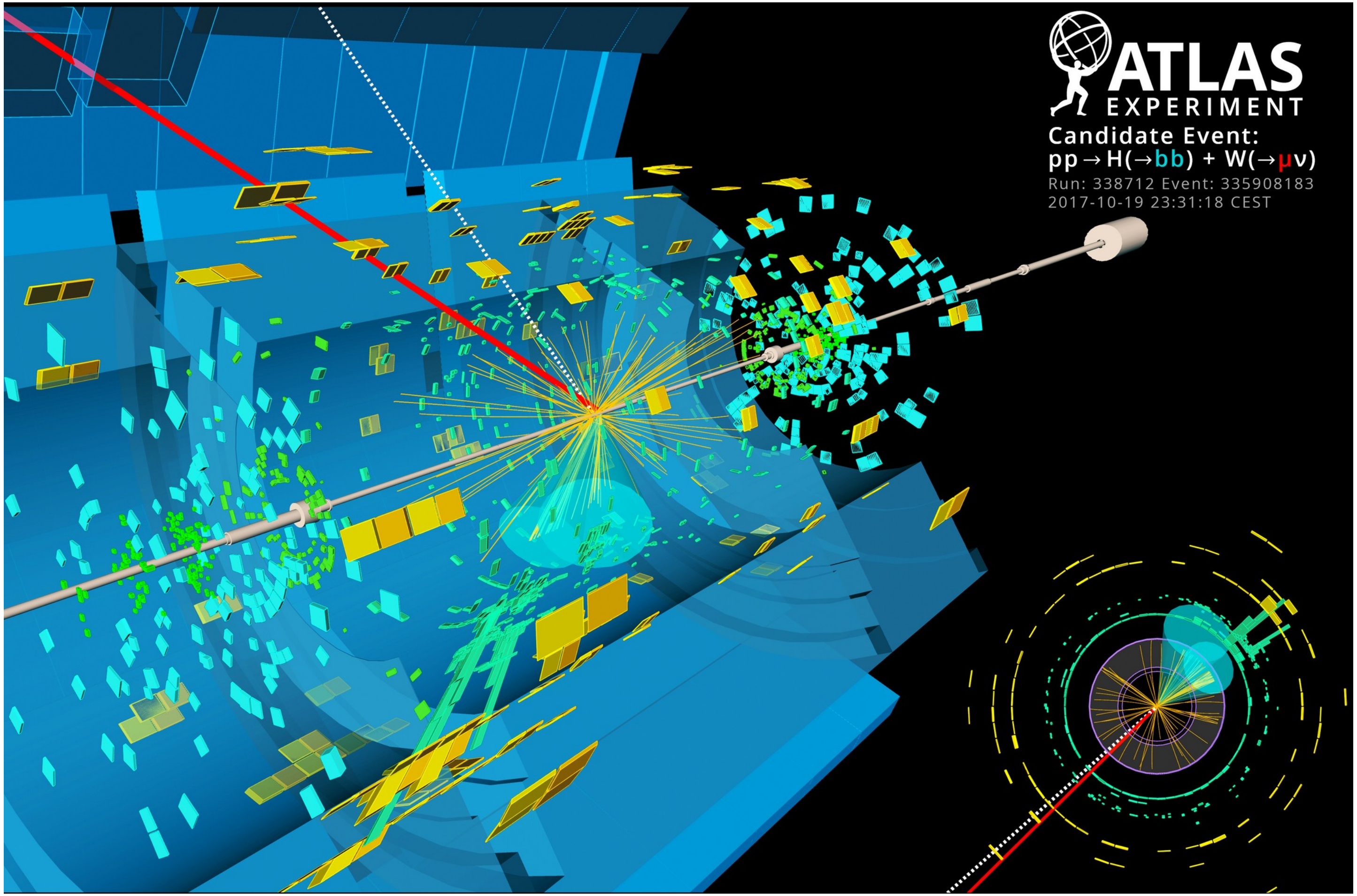
Credit: CERN



Theory/ Standard Model



Collision events



Very rough **Theory picture of hadron collision events**

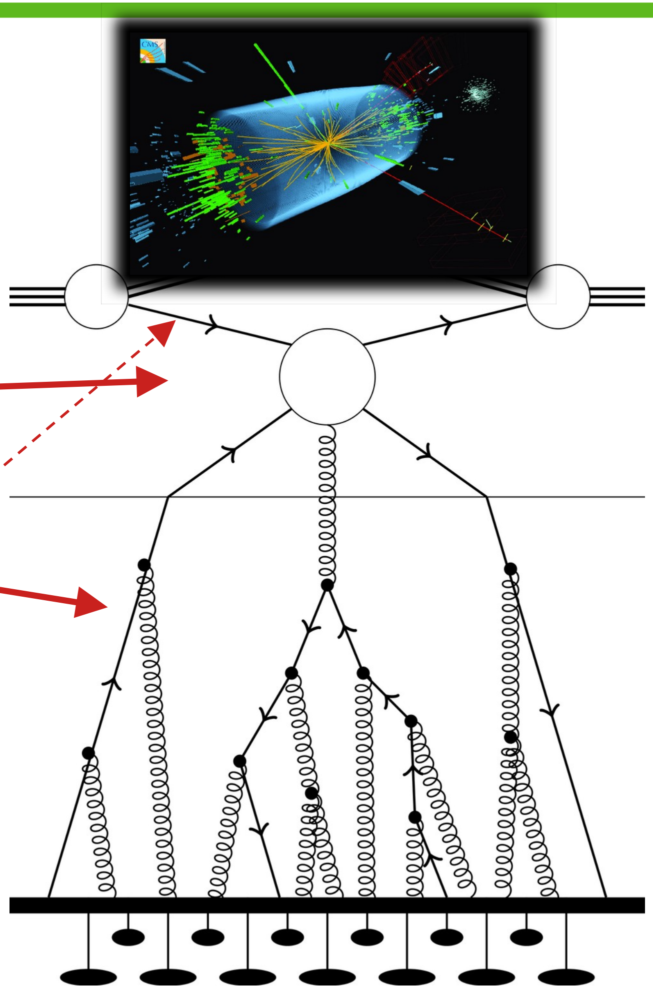
Guiding principle: factorization
"What you see depends on the energy scale"

In Quantum Chromodynamics (QCD):

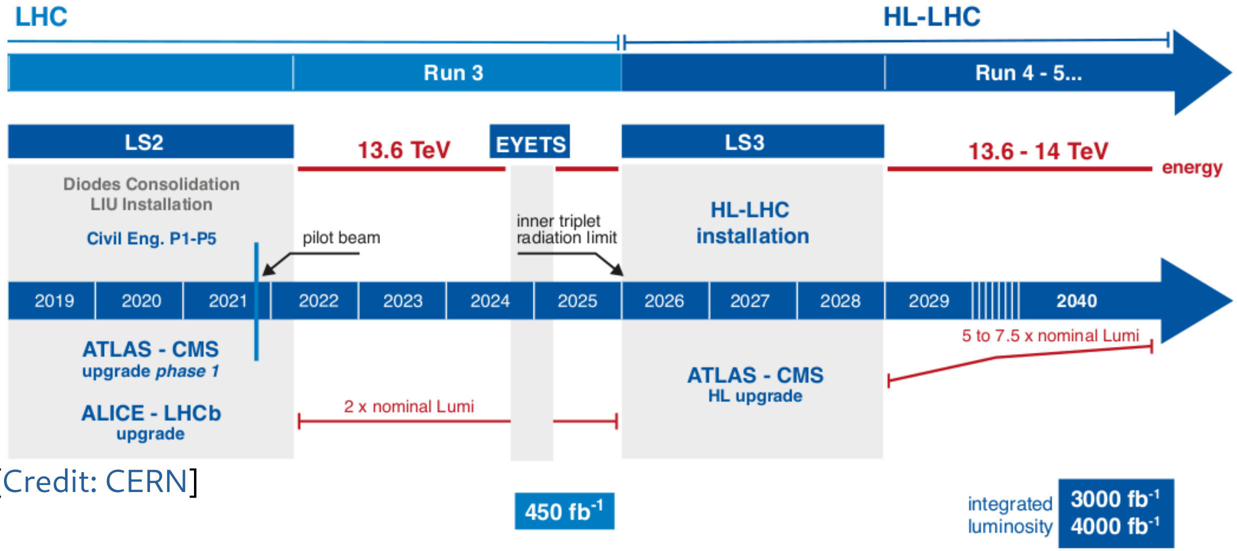
$Q \gg \Lambda_{\mathrm{QCD}}$ **Fixed-order perturbation theory**
scattering of individual partons

$Q \gtrsim \Lambda_{\mathrm{QCD}}$ **Parton-shower/Resummation**
all-order bridge between perturbative and non-perturbative physics

$Q \sim \Lambda_{\mathrm{QCD}}$ **"Hadronization"/MPI/...**
non-perturbative physics



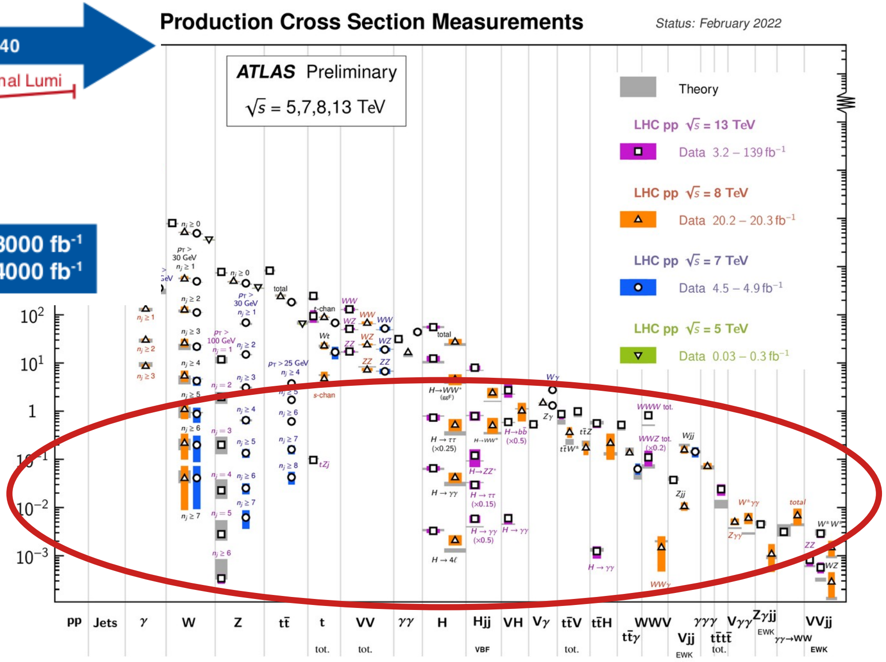
LHC Precision era and future experiments



[Credit: CERN]

→ **much more stats & hopefully similar systematics**

(difficult pile-up/radiation environment)



More precision for

→ stat. dominated, i.e. **higher multiplicity**, processes

→ cases where **systematics are statistics limited** (data-driven)

Precision predictions

Fixed order perturbation theory

→ **Core element of event simulations**
→ **describes high Q regime**

Resummation

Parton-showers

Precision theory predictions

Parametric input:
PDFs, couplings (α_s), ...

Soft physics:
MPI, colour reconnection, ...

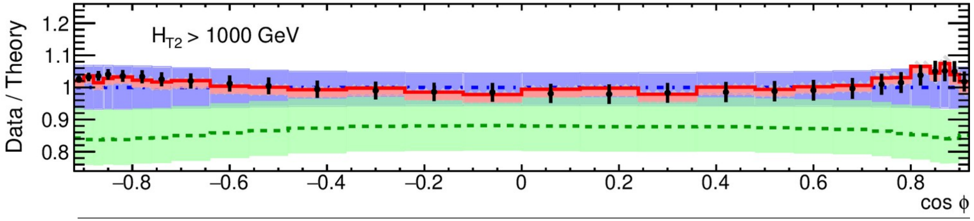
Fragmentation/hadronisation

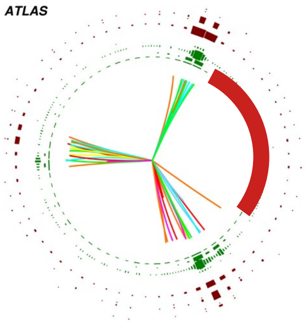
Precision through higher-order perturbation theory

Example: ATLAS multi-jet measurements

[ATLAS 2301.09351]

NNLO QCD corrections to event shapes at the LHC
Alvarez, Cantero, Czakon, Llorente, Mitov, **Poncelet** JHEP 03 (2023) 129





$\cos\phi$

Cross section = LO + NLO + NNLO + O(α_s^3)

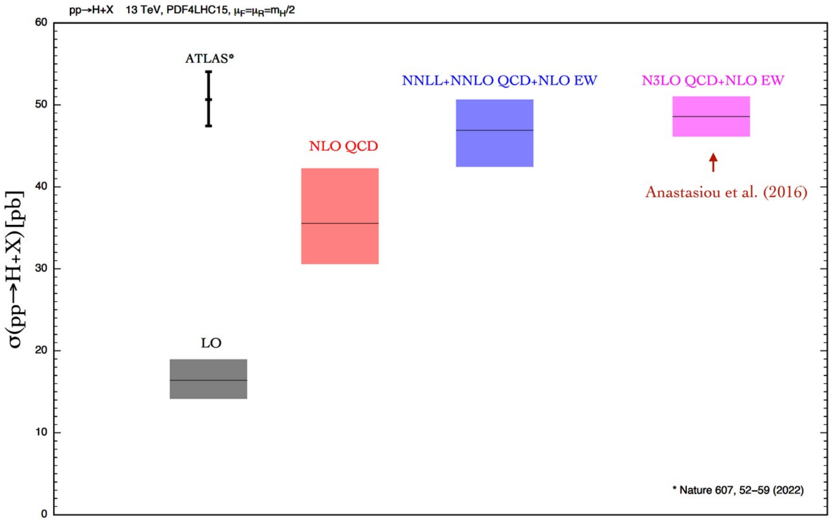
NLO $\sim (\alpha_s)^1$, NNLO $\sim (\alpha_s)^2$

Theory uncertainty: Order of magnitude | O(10%) | O(1%)

Fixed-order expansion in the strong coupling $\alpha_s(m_Z) \approx 0.118$

Experimental precision reaches percent-level already at LHC
next-to-next-to-leading order QCD needed on theory side
or even **N3LO** and beyond!

N3LO Higgs production

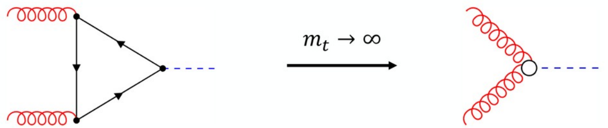


[talk by Grazzini]

- Higgs production is dominated through gluon-fusion
- Experimental measurement

$$\sigma^{\mathrm{exp.}}_{gg\to H} = 47.1 \pm 3.8 \text{ pb}$$ [CMS'22]

- HL LHC expects 2 % uncertainty
- Theory predictions need to keep up
  → Higher-order predictions crucial!

Heavy Top Limit (HTL or EFT):  $\sigma_{gg\to H} = \sigma^{\mathrm{HTL}}_{gg\to H} + \mathcal{O}\left(\frac{m_H^2}{m_t^2}\right) \quad \text{for} \quad m_t \to \infty$

Higgs Effective Field Theory (HEFT or rEFT): $\sigma^{\mathrm{N}^n\mathrm{LO}}_{\mathrm{HEFT}} = \frac{\sigma^{\mathrm{LO}}}{\sigma^{\mathrm{LO}}_{\mathrm{HTL}}}\sigma^{\mathrm{N}^n\mathrm{LO}}_{\mathrm{HTL}} \approx 1.064 \times \sigma^{\mathrm{N}^n\mathrm{LO}}_{\mathrm{HTL}}$

Precision predictions for Higgs production in gluon-fusion

[LHCH(XS)WG YR4' 16]

Immense community effort to achieve precise theory predictions

$$\sigma = 48.58\,\mathrm{pb}\,\boxed{{}^{+2.22\,\mathrm{pb}\,(+4.56\%)}_{-3.27\,\mathrm{pb}\,(-6.72\%)}}\,(\text{theory}) \pm 1.56\,\mathrm{pb}\,(3.20\%)\,(\text{PDF}+\alpha_s)\,.$$

$48.58\,\mathrm{pb} =$	$16.00\,\mathrm{pb}$	$(+32.9\%)$	(LO, rEFT)	[Georgi, Glashow, Machacek, Nanopoulos'78]
	$+\,20.84\,\mathrm{pb}$	$(+42.9\%)$	(NLO, rEFT)	[Dawson '91][Djouadi, Spira Zerwas '91]
	$-\;\;2.05\,\mathrm{pb}$	(-4.2%)	$((t,b,c)$, exact NLO)	[Graudenz, Spira, Zerwas '93]
	$+\;\;9.56\,\mathrm{pb}$	$(+19.7\%)$	(NNLO, rEFT)	[Ravindran, Smith, Van Neerven '02] [Harlander, Kilgore '02][Anastasiou, Melnikov '02]
	$+\;\;0.34\,\mathrm{pb}$	$(+0.7\%)$	(NNLO, $1/m_t$)	[Harlander, Ozeren'09][Pak, Rogal, Steinhauser'10] [Harlander, Mantler, Marzani, Ozeren '10]
	$+\;\;2.40\,\mathrm{pb}$	$(+4.9\%)$	(EW, QCD-EW)	[Aglietti, Bonciani, Degrassi,Vicini'04] [Actis, Passarino, Sturm, Uccirati'08] [Anastasiou, Boughezal, Petriello'09]
	$+\;\;1.49\,\mathrm{pb}$	$(+3.1\%)$	($\mathrm{N^3LO}$, rEFT)	[Anastasiou, Duhr, Dulat, Herzog, Mistlberger'15]

Remaining theory uncertainties

[LHCH(XS)WG YR4' 16]

N4LO approximation
[Das, Moch, Vogt '20]

aN3LO PDFs
[MSHT'22,NNPDF'24]

Exact top-mass dependence through NNLO QCD
[Czakon, Harlander, Klappert, Niggetiedt'21]

Input parameters

$\sqrt{S}$	13 TeV
m_h	125 GeV
PDF	PDF4LHC15_nnlo_100
$\alpha_s(m_Z)$	0.118
$m_t(m_t)$	162.7 GeV ($\overline{\text{MS}}$)
$m_b(m_b)$	4.18 GeV ($\overline{\text{MS}}$)
$m_c(3GeV)$	0.986 GeV ($\overline{\text{MS}}$)
$\mu = \mu_R = \mu_F$	62.5 GeV ($= m_H/2$)

δ(scale)	δ(trunc)	δ(PDF-TH)	δ(EW)	$\delta(t, b, c)$	$\delta(1/m_t)$
$+0.10$ pb -1.15 pb	± 0.18 pb	± 0.56 pb	± 0.49 pb	± 0.40 pb	± 0.49 pb
$+0.21\%$ -2.37%	$\pm 0.37\%$	$\pm 1.16\%$	$\pm 1\%$	$\pm 0.83\%$	$\pm 1\%$

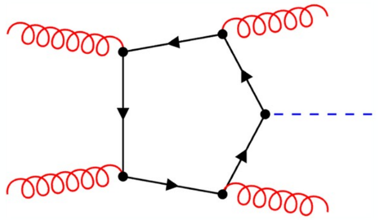
N3LO HEFT
[Mistlberger'18]

Improved QCD-EW predictions
[Bonetti, Melnikov, Trancredi'18] [Anastasiou et al '19]
[Bonetti et al. '20][Bechetti et al. '21] [Bonetti, Panzer, Trancredi '22]

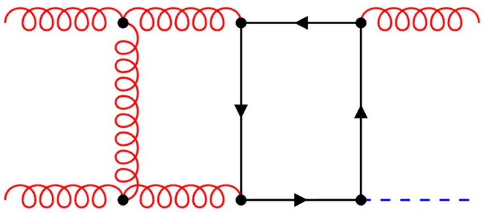
Bottom-top-interference
[Czakon, Eschment, Niggetiedt, Poncelet, Schellenberger, Phys.Rev.Lett. 132 (2024) 21, 211902, JHEP 10 (2024) 210, EurekAlert]

Bottom-top interference effects through NNLO QCD

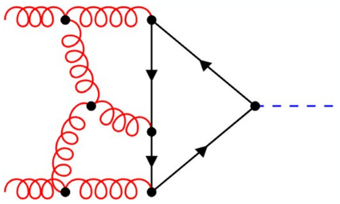
Double real (one-loop)



Real virtual (two-loop)



Double virtual (three-loop)



Renorm. scheme	$\overline{\text{MS}}$	on-shell
$\mathcal{O}(\alpha_s^2)$	-1.11	-1.98
LO	$-1.11^{+0.28}_{-0.43}$	$-1.98^{+0.38}_{-0.53}$
$\mathcal{O}(\alpha_s^3)$	-0.65	-0.44
NLO	$-1.76^{+0.27}_{-0.28}$	$-2.42^{+0.19}_{-0.12}$
$\mathcal{O}(\alpha_s^4)$	$+0.02$	$+0.43$
NNLO	$-1.74(2)^{+0.13}_{-0.03}$	$-1.99(2)^{+0.29}_{-0.15}$

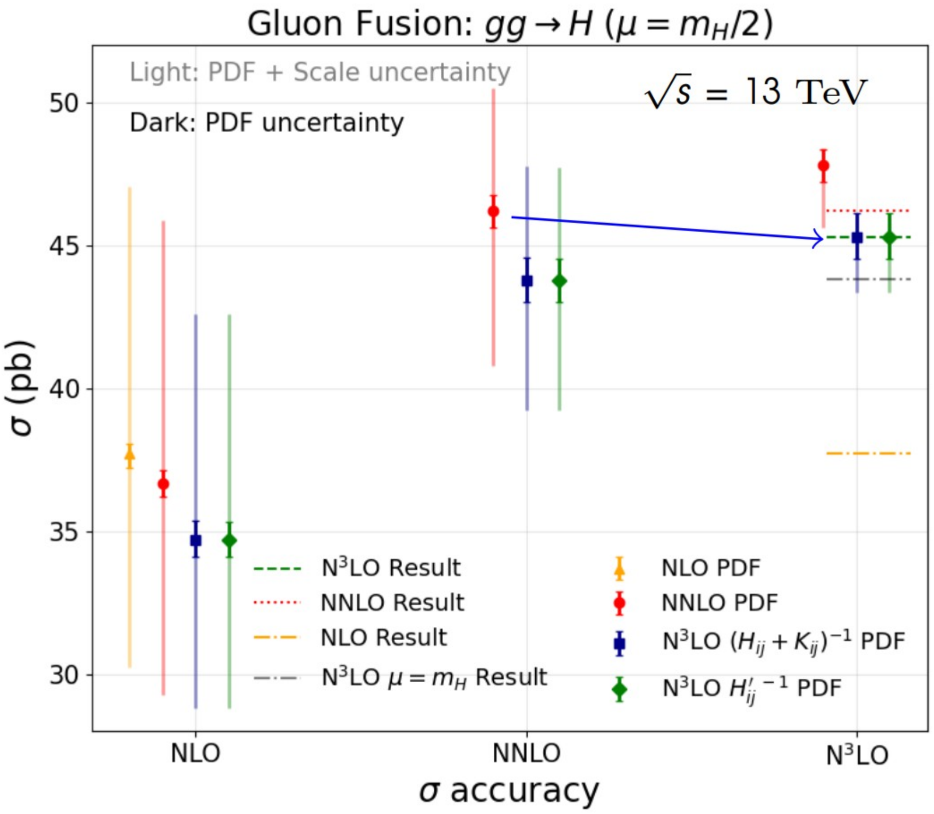
Renormalisation scheme independence at NNLO

Pure top-quark mass effects

Order	σ_{HEFT} [pb]	$(\sigma_t - \sigma_{\text{HEFT}})$ [pb]
$\mathcal{O}(\alpha_s^2)$	$+16.30$	–
LO	$16.30^{+4.36}_{-3.10}$	–
$\mathcal{O}(\alpha_s^3)$	$+21.14$	-0.303
NLO	$37.44^{+8.42}_{-6.29}$	$-0.303^{+0.10}_{-0.17}$
$\mathcal{O}(\alpha_s^4)$	$+9.72$	$+0.147(1)$
NNLO	$47.16^{+4.21}_{-4.77}$	$-0.156(1)^{+0.13}_{-0.03}$

Bottom-top interference larger than top mass effect

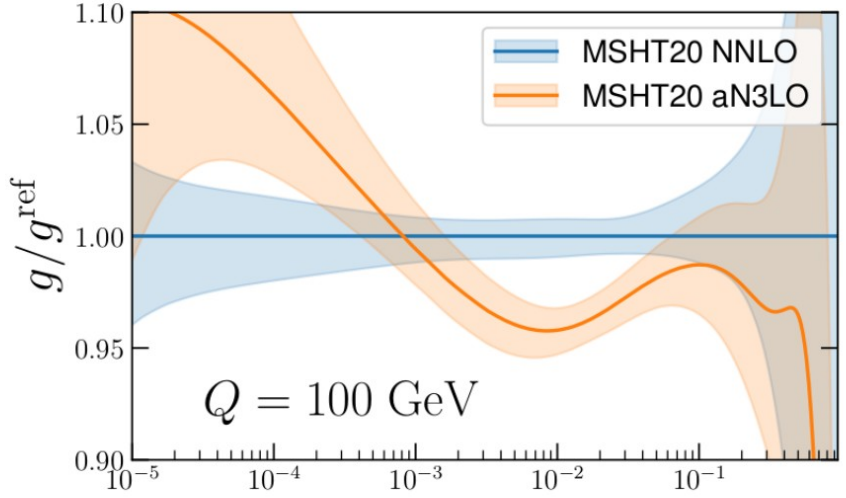
What about the PDFs?!



aN3LO PDFs:

→ approximate N3LO ingredients (splitting functions)

→ higher-order coefficient functions modelled by theory unc.



Les Houches Wish List: where do we stand?

Higgs

process	known
$pp \to H$	$\mathrm{N^3LO_{HTL}}$ $\mathrm{NNLO_{QCD}^{(t,t\times b)}}$ $\mathrm{N^{(1,1)}LO_{QCD\otimes EW}^{(HTL)}}$ $\mathrm{NLO_{QCD}}$
$pp \to H+j$	$\mathrm{NNLO_{HTL}}$ $\mathrm{NLO_{QCD}}$ $\mathrm{N^{(1,1)}LO_{QCD\otimes EW}}$
$pp \to H+2j$	$\mathrm{NLO_{HTL} \otimes LO_{QCD}}$ $\mathrm{N^3LO_{QCD}^{(VBF^*)}}$ (incl.) $\mathrm{NNLO_{QCD}^{(VBF^*)}}$ $\mathrm{NLO_{EW}^{(VBF)}}$
$pp \to H+3j$	$\mathrm{NLO_{HTL}}$ $\mathrm{NLO_{QCD}^{(VBF)}}$
$pp \to VH$	$\mathrm{N^3LO_{QCD}}$ (incl.)+ $\mathrm{NLO_{EW}}$ $\mathrm{NLO_{gg\to HZ}^{(t,b)}}$
$pp \to VH+j$	$\mathrm{NNLO_{QCD}}$ $\mathrm{NLO_{QCD} + NLO_{EW}}$
$pp \to HH$	$\mathrm{N^3LO_{HTL} \otimes NLO_{QCD}}$ $\mathrm{NLO_{EW}}$
$pp \to HH+2j$	$\mathrm{N^3LO_{QCD}^{(VBF^*)}}$ (incl.) $\mathrm{NNLO_{QCD}^{(VBF^*)}}$ $\mathrm{NLO_{EW}^{(VBF)}}$
$pp \to HHH$	$\mathrm{NNLO_{HTL}}$
$pp \to H+t\bar{t}$	$\mathrm{NLO_{QCD} + NLO_{EW}}$ $\mathrm{NNLO_{QCD}}$ (approx.)
$pp \to H+t/\bar{t}$	$\mathrm{NLO_{QCD} + NLO_{EW}}$

Vector-bosons

process	known
$pp \to V$	$\mathrm{N^3LO_{QCD} + N^{(1,1)}LO_{QCD\otimes EW}}$ $\mathrm{NLO_{EW}}$
$pp \to VV'$	$\mathrm{NNLO_{QCD} + NLO_{EW}}$ + Full $\mathrm{NLO_{QCD}}$ $(gg \to ZZ)$, approx. $\mathrm{NLO_{QCD}}$ $(gg \to WW)$
$pp \to V+j$	$\mathrm{NNLO_{QCD} + NLO_{EW}}$
$pp \to V+2j$	$\mathrm{NLO_{QCD} + NLO_{EW}}$ (QCD component) $\mathrm{NLO_{QCD} + NLO_{EW}}$ (EW component)
$pp \to V+b\bar{b}$	$\mathrm{NLO_{QCD}}$
$pp \to W+b\bar{b}$	$\mathrm{NNLO_{QCD}}$
$pp \to VV'+1j$	$\mathrm{NLO_{QCD} + NLO_{EW}}$
$pp \to VV'+2j$	$\mathrm{NLO_{QCD}}$ (QCD component) $\mathrm{NLO_{QCD} + NLO_{EW}}$ (EW component)
$pp \to W^+W^+ + 2j$	Full $\mathrm{NLO_{QCD} + NLO_{EW}}$
$pp \to W^+W^- + 2j$	$\mathrm{NLO_{QCD} + NLO_{EW}}$ (EW component)
$pp \to W^+Z + 2j$	$\mathrm{NLO_{QCD} + NLO_{EW}}$ (EW component)
$pp \to ZZ + 2j$	Full $\mathrm{NLO_{QCD} + NLO_{EW}}$
$pp \to VV'V''$	$\mathrm{NLO_{QCD}}$ +$\mathrm{NLO_{EW}}$ (w/ decays)
$pp \to WWW$	$\mathrm{NLO_{QCD}}$ + $\mathrm{NLO_{EW}}$ (off-shell)
$pp \to W^+W^+(V \to jj)$	$\mathrm{NLO_{QCD}}$ + $\mathrm{NLO_{EW}}$ (off-shell)
$pp \to WZ(V \to jj)$	$\mathrm{NLO_{QCD}}$ + $\mathrm{NLO_{EW}}$ (off-shell)
$pp \to \gamma\gamma$	$\mathrm{NNLO_{QCD} + NLO_{EW}}$
$pp \to \gamma + j$	$\mathrm{NNLO_{QCD} + NLO_{EW}}$
$pp \to \gamma\gamma + j$	$\mathrm{NNLO_{QCD} + NLO_{EW}}$ + $\mathrm{NLO_{QCD}}$ (gg channel)
$pp \to \gamma\gamma\gamma$	$\mathrm{NNLO_{QCD}}$

Top-quarks

process	known
$pp \to t\bar{t}$	$\mathrm{NNLO_{QCD} + NLO_{EW}}$ (w/o decays) $\mathrm{NLO_{QCD} + NLO_{EW}}$ (off-shell) $\mathrm{NNLO_{QCD}}$ (w/ decays)
$pp \to t\bar{t}+j$	$\mathrm{NLO_{QCD}}$ (off-shell effects) $\mathrm{NLO_{EW}}$ (w/o decays)
$pp \to t\bar{t}+2j$	$\mathrm{NLO_{QCD}}$ (w/o decays)
$pp \to t\bar{t}+V'$	$\mathrm{NLO_{QCD} + NLO_{EW}}$ (w decays)
$pp \to t\bar{t}+\gamma$	$\mathrm{NLO_{QCD}}$ (off-shell)
$pp \to t\bar{t}+Z$	$\mathrm{NLO_{QCD} + NLO_{EW}}$ (off-shell)
$pp \to t\bar{t}+W$	$\mathrm{NLO_{QCD} + NLO_{EW}}$ (off-shell)
$pp \to t/\bar{t}$	$\mathrm{NNLO_{QCD}}$*(w decays) $\mathrm{NLO_{EW}}$ (w/o decays)
$pp \to tZj$	$\mathrm{NLO_{QCD} + NLO_{EW}}$ (off shell)
$pp \to t\bar{t}t\bar{t}$	$\mathrm{NLO_{QCD}}$ (w decay) $\mathrm{NLO_{EW}}$ (w/o decays)

Jets

process	known
$pp \to 2$ jets	$\mathrm{NNLO_{QCD}}$ $\mathrm{NLO_{QCD} + NLO_{EW}}$
$pp \to 3$ jets	$\mathrm{NNLO_{QCD} + NLO_{EW}}$

NLO

NNLO

N3LO

[LHWL23]

Les Houches Wish List: where do we want to go?

Higgs

process	desired
$pp \to H$	$\mathrm{N^4LO_{HTL}}$ (incl.)
$pp \to H + j$	$\mathrm{NNLO_{HTL}} \otimes \mathrm{NLO_{QCD}} + \mathrm{NLO_{EW}}$ $\mathrm{N^3LO_{HTL}}$ $\mathrm{NNLO_{QCD}}$
$pp \to H + 2j$	$\mathrm{NNLO_{HTL}} \otimes \mathrm{NLO_{QCD}} + \mathrm{NLO_{EW}}$ $\mathrm{N^3LO_{QCD}^{(VBF^*)}}$ $\mathrm{NNLO_{QCD}^{(VBF)}}$ $\mathrm{NLO_{QCD}}$
$pp \to H + 3j$	$\mathrm{NLO_{QCD}} + \mathrm{NLO_{EW}}$ $\mathrm{NNLO_{QCD}^{(VBF^*)}}$
$pp \to VH$	$\mathrm{N^3LO_{QCD}}$ $\mathrm{N^{(1,1)}LO_{QCD \otimes EW}}$
$pp \to VH + j$	
$pp \to HH$	$\mathrm{NNLO_{QCD}}$
$pp \to HH + 2j$	$\mathrm{NLO_{QCD}}$
$pp \to HHH$	$\mathrm{NLO_{QCD}}$
$pp \to H + t\bar{t}$	$\mathrm{NNLO_{QCD}}$
$pp \to H + t/\bar{t}$	$\mathrm{NNLO_{QCD}}$

Vector-bosons

process	desired
$pp \to V$	$\mathrm{N^2LO_{EW}}$
$pp \to VV'$	Full $\mathrm{NLO_{QCD}}$ (gg channel, w/ massive loops) $\mathrm{N^{(1,1)}LO_{QCD \otimes EW}}$
$pp \to V + j$	hadronic decays
$pp \to V + 2j$	$\mathrm{NNLO_{QCD}}$
$pp \to V + b\bar{b}$ $pp \to W + b\bar{b}$	$\mathrm{NNLO_{QCD}} + \mathrm{NLO_{EW}}$
$pp \to VV' + 1j$	$\mathrm{NNLO_{QCD}}$
$pp \to VV' + 2j$ $pp \to W^+W^+ + 2j$ $pp \to W^+W^- + 2j$ $pp \to W^+Z + 2j$ $pp \to ZZ + 2j$	Full $\mathrm{NLO_{QCD}} + \mathrm{NLO_{EW}}$
$pp \to VV'V''$ $pp \to WWW$ $pp \to W^+W^+(V \to jj)$ $pp \to WZ(V \to jj)$	$\mathrm{NLO_{QCD}} + \mathrm{NLO_{EW}}$ (off-shell)
$pp \to \gamma\gamma$	$\mathrm{N^3LO_{QCD}}$
$pp \to \gamma + j$	$\mathrm{N^3LO_{QCD}}$
$pp \to \gamma\gamma + j$	
$pp \to \gamma\gamma\gamma$	$\mathrm{NLO_{EW}}$

Top-quarks

process	desired
$pp \to t\bar{t}$	$\mathrm{N^3LO_{QCD}}$
$pp \to t\bar{t} + j$	$\mathrm{NNLO_{QCD}} + \mathrm{NLO_{EW}}$ (w decays)
$pp \to t\bar{t} + 2j$	$\mathrm{NLO_{QCD}} + \mathrm{NLO_{EW}}$ (w decays)
$pp \to t\bar{t} + V'$ $pp \to t\bar{t} + \gamma$ $pp \to t\bar{t} + Z$ $pp \to t\bar{t} + W$	$\mathrm{NNLO_{QCD}} + \mathrm{NLO_{EW}}$ (w decays)
$pp \to t/\bar{t}$	$\mathrm{NNLO_{QCD}} + \mathrm{NLO_{EW}}$ (w decays)
$pp \to tZj$	$\mathrm{NNLO_{QCD}} + \mathrm{NLO_{EW}}$ (w/o decays)
$pp \to t\bar{t}t\bar{t}$	$\mathrm{NLO_{QCD}} + \mathrm{NLO_{EW}}$ (off-shell) $\mathrm{NNLO_{QCD}}$

Jets

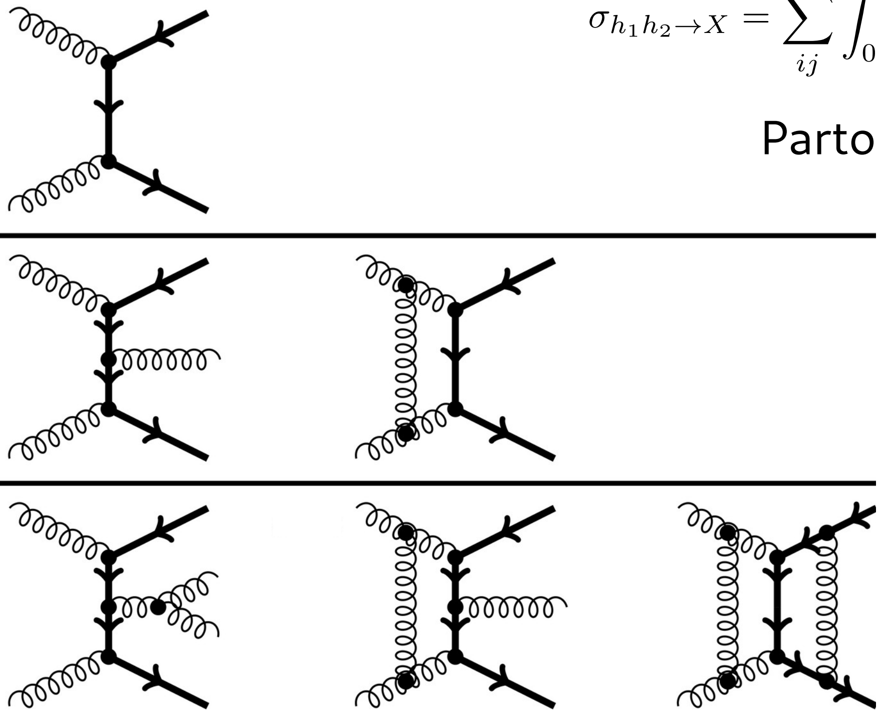
process	desired
$pp \to 2\,\mathrm{jets}$	$\mathrm{N^3LO_{QCD}} + \mathrm{NLO_{EW}}$
$pp \to 3\,\mathrm{jets}$	

NNLO QCD in collinear factorization

LO

NLO

NNLO



$$\sigma_{h_1 h_2 \to X} = \sum_{ij} \int_0^1 \int_0^1 \mathrm{d}x_1 \mathrm{d}x_2 \phi_{i,h_1}(x_1, \mu_F^2) \phi_{j,h_2}(x_2, \mu_F^2) \hat{\sigma}_{ij \to X}(\alpha_s(\mu_R^2), \mu_R^2, \mu_F^2)$$

Parton distribution functions

Partonic cross section:

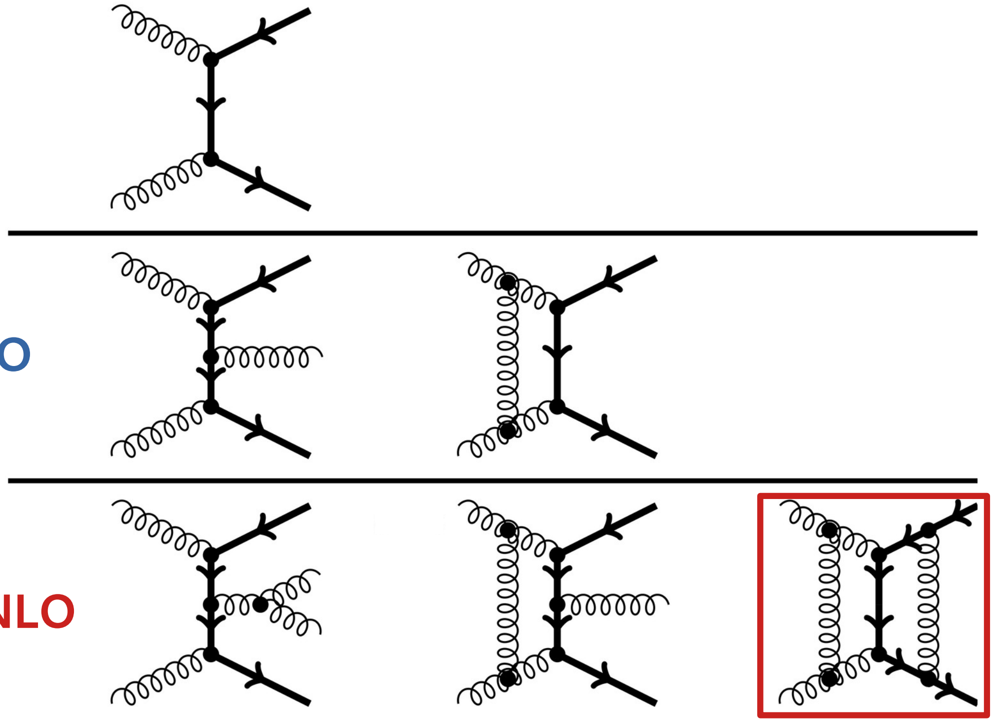
$$\hat{\sigma}_{ab \to X} = \hat{\sigma}^{(0)}_{ab \to X} + \hat{\sigma}^{(1)}_{ab \to X} + \hat{\sigma}^{(2)}_{ab \to X} + \mathcal{O}(\alpha_s^3)$$

NNLO QCD challenges: two-loop amplitudes

LO

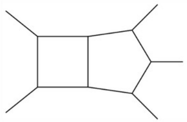
NLO

NNLO

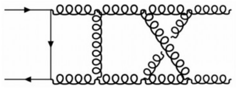


How to compute
multi-scale two-loop QCD amplitudes?
→ **fast growing complexity:**
rational coef. and **special functions**
→ deeper understanding of the analytical properties
→ refinement of computational tools

Two-loop 5-point
[Abreu, Agarwal, Badger, Buccioni, Chawdhry, Chicherin,Czakon, de Laurenties, Febres-Cordero, Gambuti, Gehrmann, Henn, Lo Presti, Manteuffel, Ma, Mitov, Page, Peraro, Poncelet, Schabiner Sotnikov, Tancredi, Zhang,...]

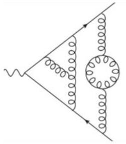


Three-loop 4-point

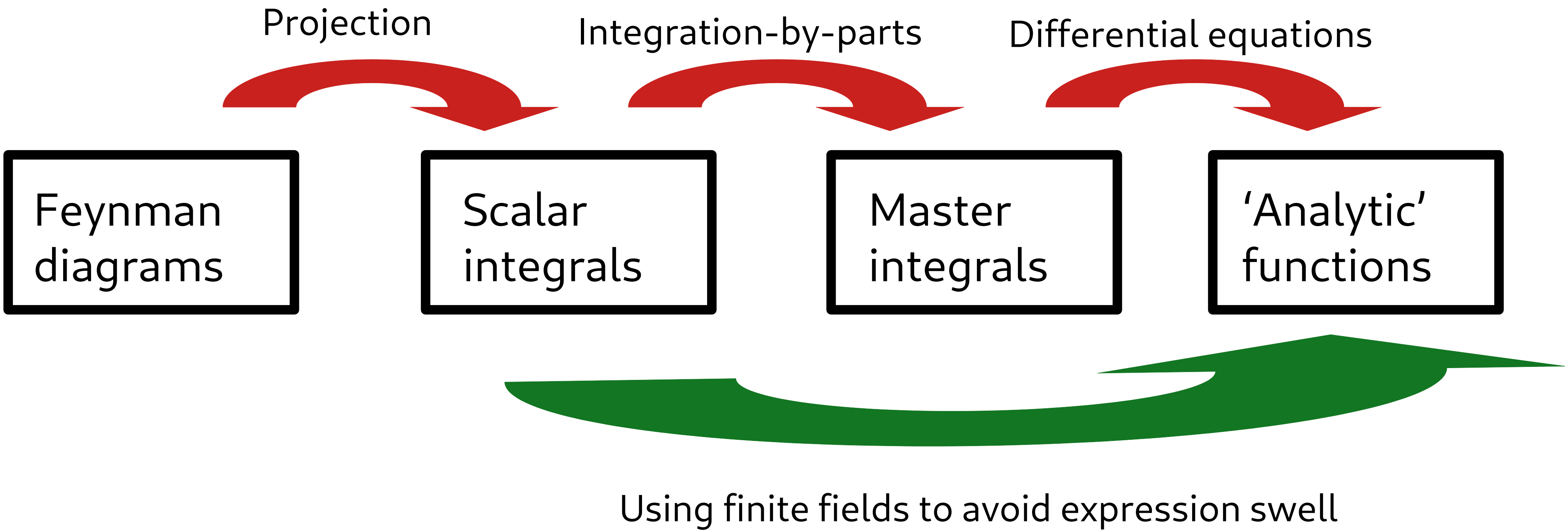


[Bargiela, Dobadilla, Canko, Caola, Jakubcik, Gambuti, Gehrmann, Henn, Lim, Mella, Mistleberger, Wasser, Manteuffel, Syrrakow, Smirnov, Trancredi, ...]

Four-loop 3-point
[Henn, Lee, Manteuffel, Schabinger, Smirnov, Steinhauser,...]



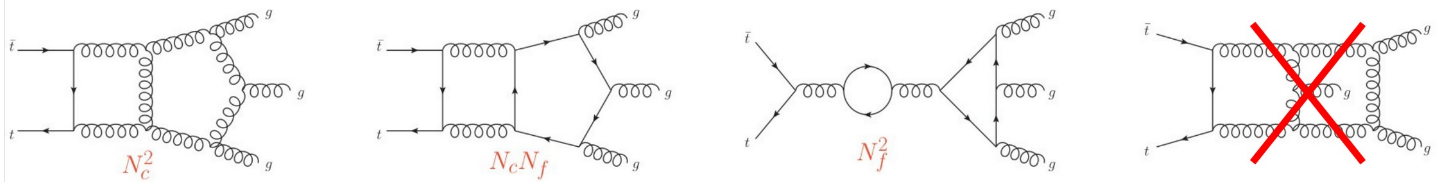
Approach to multi-loop amplitudes



Virtual amplitudes

Double virtual QCD corrections to tt̄+jet production at the LHC,
Badger, Becchetti, Brancaccio, Czakon, Hartanto, **Poncelet**, Zoia
[arxiv:2511.11424]

Sample diagrams



Decomposition:

Colour structures

$$\mathcal{M}^{(L)}\left(1_{\bar{t}}, 2_t, 3_g, 4_g, 5_g\right) = \sqrt{2}\,\bar{g}_s^3 \left[(4\pi)^{\epsilon} e^{-\epsilon\gamma_E} \frac{\bar{\alpha}_s}{4\pi}\right]^L \times \sum_{\sigma\in Z_3} \left(t^{a_{\sigma(3)}} t^{a_{\sigma(4)}} t^{a_{\sigma(5)}}\right)_{i_2}^{\bar{i}_1} \mathcal{A}_g^{(L)}\left(1_{\bar{t}}, 2_t, \sigma(3)_g, \sigma(4)_g, \sigma(5)_g\right)$$

Leading colour expansion of partial amplitudes:

$$\mathcal{F}_x^{(1)} = N_c\, F_x^{(1),N_c} + n_f\, F_x^{(1),n_f}\,,$$

$$\mathcal{F}_x^{(2)} = N_c^2\, F_x^{(2),N_c^2} + N_c n_f\, F_x^{(2),N_c n_f} + n_f^2\, F_x^{(2),n_f^2}\,.$$

scalar&finite functions

Spin structure:

$$\mathcal{A}_x^{(L)h_3h_4h_5} = \sum_{i=1}^{4} \Psi_i\, \mathcal{G}_{x;i}^{(L)h_3h_4h_5} \qquad \mathcal{G}_{x;i}^{(L)h_3h_4h_5} = \sum_{j=1}^{4} \left(\Omega^{-1}\right)_{ij} \tilde{\mathcal{A}}_{x;j}^{(L)h_3h_4h_5} \xrightarrow{\text{IR renorm.}} \tilde{\mathcal{F}}_{x;i}^{(L)h_3h_4h_5}$$

Integration-By-Parts reduction

$$\tilde{\mathcal{F}}^{(L)h_1h_2h_3}_{x,i} = \sum_j c_j(\{p\},\epsilon)\mathcal{I}_j(\{p\},\epsilon)$$ → prohibitively large number of integrals

$$\mathcal{I}_i(\{p\},\epsilon) \equiv \mathcal{I}(\vec{n_i},\{p\},\epsilon) = \int \frac{\mathrm{d}^d k_1}{(2\pi)^d}\frac{\mathrm{d}^d k_2}{(2\pi)^d}\prod_{k=1} D_k^{-n_{i,k}}(\{p\},\{k\})$$

Integration-By-Parts identities connect different integrals → **system of equations**
→ only a **small number of independent "master" integrals**

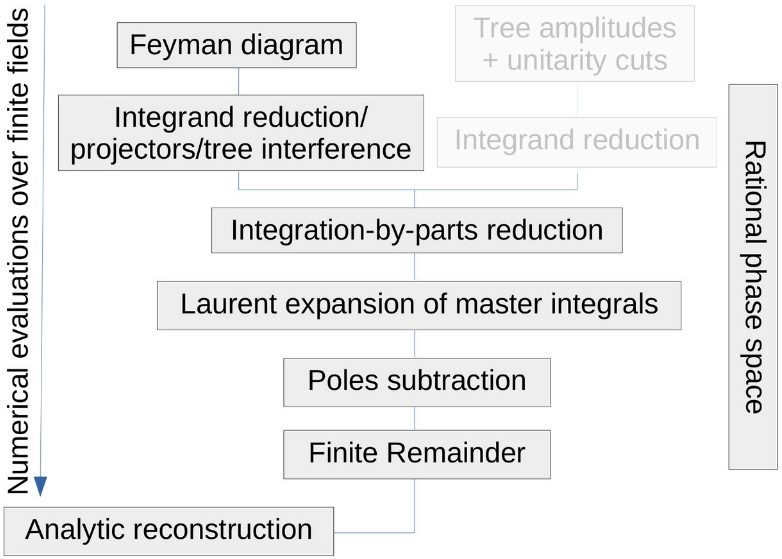
$$0 = \int \frac{\mathrm{d}^d k_1}{(2\pi)^d}\frac{\mathrm{d}^d k_2}{(2\pi)^d} l_\mu \frac{\partial}{\partial l^\mu}\prod_{k=1}^{11} D_k^{-n_{i,k}}(\{p\},\{k\}) \quad \text{with} \quad l \in \{p\}\cap\{k\}$$

$$\longrightarrow \quad \tilde{\mathcal{F}}^{(L)h_1h_2h_3}_{x,i} = \sum_j d_j(\{p\},\epsilon)\mathrm{MI}_j(\{p\},\epsilon)$$

Masters expressed in basis functions: $\tilde{\mathcal{F}}^{(L)h_3h_4h_5}_{x;i} = \sum_k r^{(L)h_3h_4h_5}_{x;i,k}\, m_k(\vec{f})$

Reconstruction of Amplitudes

Workflow



Credit: Bayu

QGRAF[Nogueira], FORM[Vermaseren,etal]
MATHEMATICA, SPINNEY[Cullen,etal]

finite field framework: FINITEFLOW[Peraro(2019)]

IBP identities generated using LITERED[Lee(2012)] solved numerically in FINITEFLOW using Laporta algorithm[Laporta(2000)]

Mature technology + new optimizations

- Syzygy and module intersection techniques to simplify IBPs [NeatIBP]
- Exploitation of Q-linear relations
- Denominator Ansaetze
- on-the-fly univariate partial fraction

	1	2	3	4
master integral coefficients of mass-renormalised amplitude (full ϵ dependence)	404/393	398/389	411/402	421/411
special function coefficients of finite remainder	314/303	305/296	321/312	326/317
linear relations	291/280	287/278	299/293	304/299
denominator matching #1	291/0	287/0	299/0	304/0
partial fraction decomposition in x_{5123}	44/40	55/51	57/54	58/54
denominator matching #2	44/0	54/0	54/0	56/0
number of sample points (1 prime field)	137076	89624	161482	179838

Master integrals & finite remainder

Differential Equations: $\mathrm{d}\vec{\mathrm{MI}} = \mathrm{d}A(\{p\},\epsilon)\vec{\mathrm{MI}}$

Canonical basis: $\mathrm{d}\vec{\mathrm{MI}} = \epsilon\mathrm{d}\tilde{A}(\{p\})\vec{\mathrm{MI}}$

[Remiddi, 97]
[Gehrmann, Remiddi, 99]
[Henn, 13]

→ not available in ttj case (elliptic integrals):
However, still translates into a system of DEQs for basis functions: $\mathrm{d}G(\vec{d}) = M(\vec{d}) \cdot G(\vec{d})$

$$G(\vec{d}) = \begin{pmatrix} f_i^{(4^*)} \\ f_i^{(4)} \\ \text{weight-3} \\ \text{weight-2} \\ f_i^{(1)} \\ 1 \end{pmatrix}, \qquad M(\vec{d}) = \begin{pmatrix} Y_{4^*,4^*} & 0 & Y_{4^*,3} & Y_{4^*,2} & 0 & 0 \\ 0 & 0 & X_{4,3} & 0 & 0 & 0 \\ 0 & 0 & 0 & X_{3,2} & 0 & 0 \\ 0 & 0 & 0 & 0 & X_{2,1} & 0 \\ 0 & 0 & 0 & 0 & 0 & X_{1,0} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Solve numerically from pre-computed boundary points [AMFlow]

Numerical evaluation of master integrals

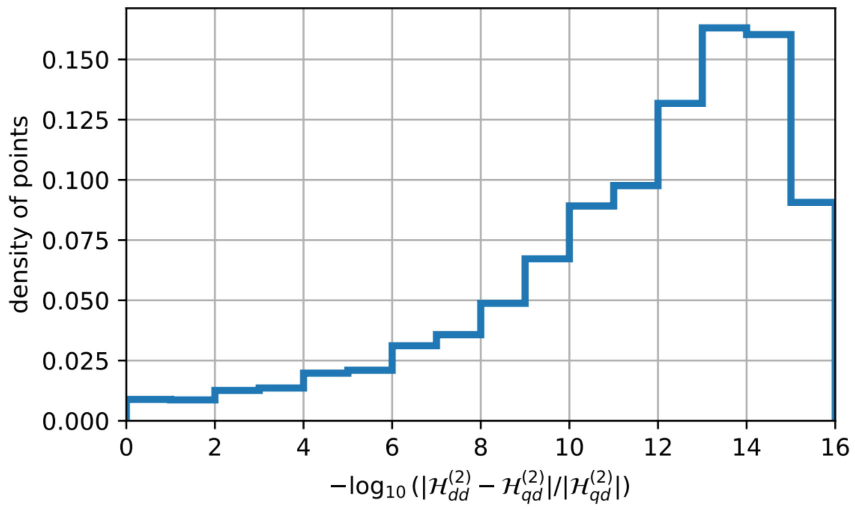
Channel	Functions [s]	Coefficents [s]	Assembly [s]	total [s]
$gg \to \bar{t}tg$	2.69	30.90	1.58	35.17
$\bar{q}g \to \bar{t}t\bar{q}$	2.16	9.40	0.18	11.74
$qg \to \bar{t}tq$	2.50	9.62	0.21	12.33
$q\bar{q} \to \bar{t}tg$	2.12	9.30	0.18	11.60

→ Solve DEQ by direct integration
(Bulirsch-Stoer algorithm)

→ Error control based on each function's contribution to squared matrix element

$$\mathcal{Q} \equiv \frac{1}{(4\pi)^4} \frac{2 \overline{\sum_{\text{colour}}} \overline{\sum_{\text{pol.}}} \mathcal{R}^{(0)*}\mathcal{R}^{(2)}}{\overline{\sum_{\text{colour}}} \overline{\sum_{\text{pol.}}} \left|\mathcal{R}^{(0)}\right|^2}$$

$$\Delta\mathcal{Q}(t) = \left[\sum_{i=1}^{947} \left| \frac{\partial Q}{\partial f_i} \right|^2_{\vec{f}=\vec{f}(t)} \left|\Delta f_i(t)\right|^2 \right]^{1/2}$$



→ directly control numerical error of the final result

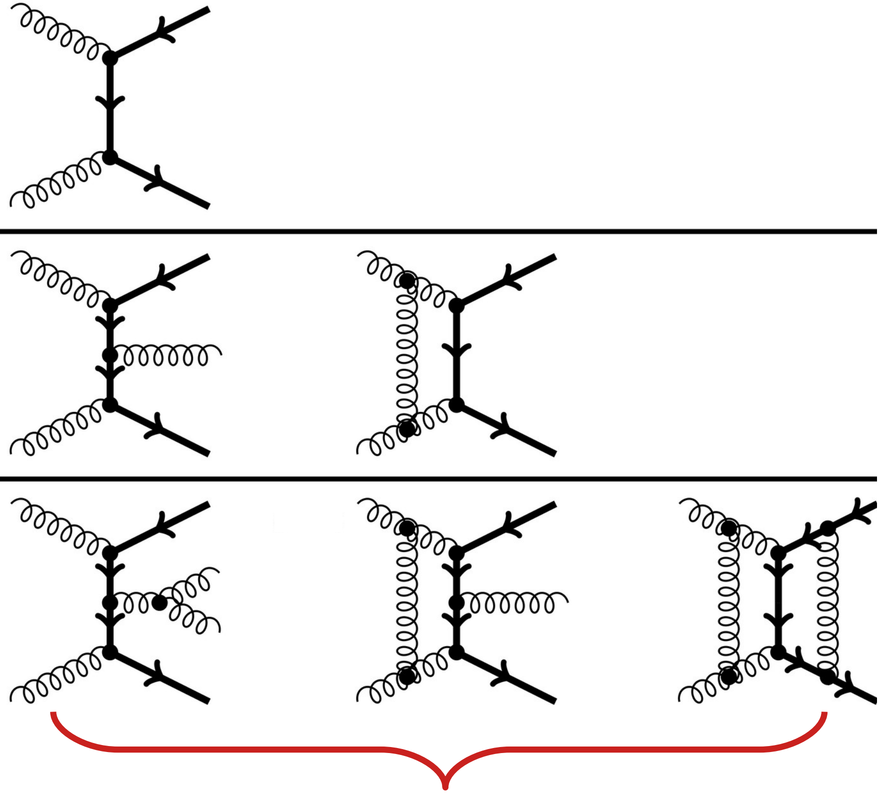
→ ready for phenomenology

NNLO QCD challenges: real radiation

LO

NLO

NNLO



IR-finite cross section

How to achieve **infrared (IR) finite differential** cross sections at NNLO QCD?

~20 years to solve this problem

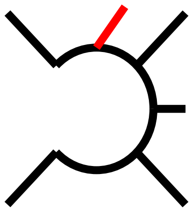
Next-to-leading order case

$$\hat{\sigma}^{(1)}_{ab} = \hat{\sigma}^{\mathrm{R}}_{ab} + \hat{\sigma}^{\mathrm{V}}_{ab} + \hat{\sigma}^{\mathrm{C}}_{ab}$$

Kinoshita-Lee-Nauenberg (KLN) theorem
sum is finite for sufficiently inclusive observables and regularization scheme independent

Each term separately infrared (IR) divergent:

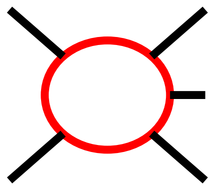
Real corrections:



$$\hat{\sigma}^{\mathrm{R}}_{ab} = \frac{1}{2\hat{s}} \int \mathrm{d}\Phi_{n+1} \left\langle \mathcal{M}^{(0)}_{n+1} \middle| \mathcal{M}^{(0)}_{n+1} \right\rangle \mathrm{F}_{n+1}$$

Phase space integration over unresolved configurations

Virtual corrections:



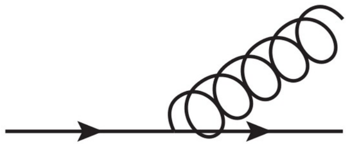
$$\hat{\sigma}^{\mathrm{V}}_{ab} = \frac{1}{2\hat{s}} \int \mathrm{d}\Phi_{n}\, 2\mathrm{Re} \left\langle \mathcal{M}^{(0)}_{n} \middle| \mathcal{M}^{(1)}_{n} \right\rangle \mathrm{F}_{n}$$

Integration over loop-momentum
(UV divergences cured by renormalization)

IR singularities in real radiation

$$\hat{\sigma}^{\mathrm{R}}_{ab} = \frac{1}{2\hat{s}} \int \mathrm{d}\Phi_{n+1} \left\langle \mathcal{M}^{(0)}_{n+1} \middle| \mathcal{M}^{(0)}_{n+1} \right\rangle \mathrm{F}_{n+1}$$

Finite function

$\ni \mathrm{d}\Phi_1$ 

$$\sim \int_0 \mathrm{d}E\mathrm{d}\theta \frac{1}{E(1-\cos\theta)} f(E, \cos(\theta))$$

Divergent

Regularization in Conventional Dimensional Regularization (CDR) $d = 4 - 2\epsilon$

$$\to \int_0 \mathrm{d}E\mathrm{d}\theta \frac{1}{E^{1-2\epsilon}(1-\cos\theta)^{1-\epsilon}} f(E, \cos(\theta)) \sim \frac{1}{\epsilon^2} + \dots$$

Cancellation against similar divergences in

$$\hat{\sigma}^{\mathrm{V}}_{ab} = \frac{1}{2\hat{s}} \int \mathrm{d}\Phi_n \, 2\mathrm{Re} \left\langle \mathcal{M}^{(0)}_{n} \middle| \mathcal{M}^{(1)}_{n} \right\rangle \mathrm{F}_{n}$$

How to extract these poles? Slicing and Subtraction

Central idea: Divergences arise from infrared (IR, soft/collinear) limits → Factorization!

Slicing

$$\hat{\sigma}^{\mathrm{R}}_{ab} = \frac{1}{2\hat{s}} \int_{\delta(\Phi)\geq\delta_c} \mathrm{d}\Phi_{n+1} \left\langle \mathcal{M}^{(0)}_{n+1} \middle| \mathcal{M}^{(0)}_{n+1} \right\rangle \mathrm{F}_{n+1} + \frac{1}{2\hat{s}} \int_{\delta(\Phi)<\delta_c} \mathrm{d}\Phi_{n+1} \left\langle \mathcal{M}^{(0)}_{n+1} \middle| \mathcal{M}^{(0)}_{n+1} \right\rangle \mathrm{F}_{n+1}$$

$$\approx \frac{1}{2\hat{s}} \int_{\delta(\Phi)\geq\delta_c} \mathrm{d}\Phi_{n+1} \left\langle \mathcal{M}^{(0)}_{n+1} \middle| \mathcal{M}^{(0)}_{n+1} \right\rangle \mathrm{F}_{n+1} + \frac{1}{2\hat{s}} \int \mathrm{d}\Phi_n \, \tilde{M}(\delta_c)\mathrm{F}_n + \mathcal{O}(\delta_c)$$

$$\ldots + \hat{\sigma}^{\mathrm{V}}_{ab} = \mathrm{finite}$$

Subtraction

$$\hat{\sigma}^{\mathrm{R}}_{ab} = \frac{1}{2\hat{s}} \int \left(\mathrm{d}\Phi_{n+1} \left\langle \mathcal{M}^{(0)}_{n+1} \middle| \mathcal{M}^{(0)}_{n+1} \right\rangle \mathrm{F}_{n+1} - \mathrm{d}\tilde{\Phi}_{n+1} \, \mathcal{S}\mathrm{F}_n \right) + \frac{1}{2\hat{s}} \int \mathrm{d}\tilde{\Phi}_{n+1} \, \mathcal{S}\mathrm{F}_n$$

$$\frac{1}{2\hat{s}} \int \mathrm{d}\tilde{\Phi}_{n+1} \, \mathcal{S}\mathrm{F}_n = \frac{1}{2\hat{s}} \int \underline{\mathrm{d}\Phi_n \, \mathrm{d}\Phi_1} \, \mathcal{S}\mathrm{F}_n$$

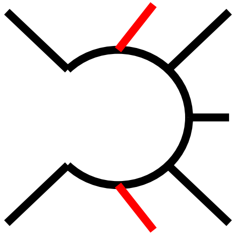
Phase space factorization
→ momentum mappings

Most popular
NLO QCD schemes:
CS [hep-ph/9605323],
FKS [hep-ph/9512328]

→ Basis of modern event simulation

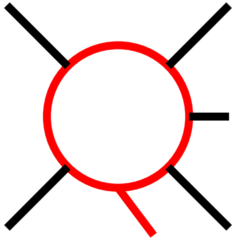
Partonic cross section beyond NLO

$$\hat{\sigma}_{ab}^{(2)} = \hat{\sigma}_{ab}^{\mathrm{VV}} + \hat{\sigma}_{ab}^{\mathrm{RV}} + \hat{\sigma}_{ab}^{\mathrm{RR}} + \hat{\sigma}_{ab}^{\mathrm{C2}} + \hat{\sigma}_{ab}^{\mathrm{C1}}$$



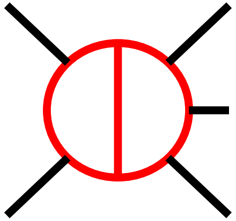
Real-Real

$$\hat{\sigma}_{ab}^{\mathrm{RR}} = \frac{1}{2\hat{s}} \int \mathrm{d}\Phi_{n+2} \left\langle \mathcal{M}_{n+2}^{(0)} \middle| \mathcal{M}_{n+2}^{(0)} \right\rangle \mathrm{F}_{n+2}$$



Real-Virtual

$$\hat{\sigma}_{ab}^{\mathrm{RV}} = \frac{1}{2\hat{s}} \int \mathrm{d}\Phi_{n+1}\, 2\mathrm{Re} \left\langle \mathcal{M}_{n+1}^{(0)} \middle| \mathcal{M}_{n+1}^{(1)} \right\rangle \mathrm{F}_{n+1}$$



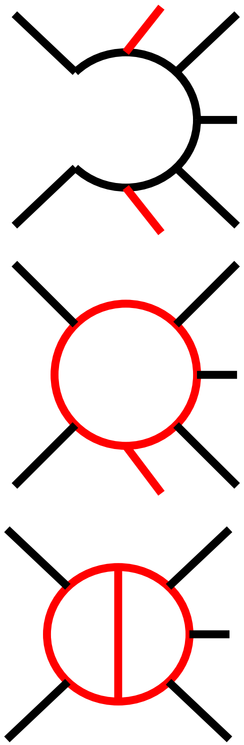
Virtual-Virtual

$$\hat{\sigma}_{ab}^{\mathrm{VV}} = \frac{1}{2\hat{s}} \int \mathrm{d}\Phi_{n} \left(2\mathrm{Re} \left\langle \mathcal{M}_{n}^{(0)} \middle| \mathcal{M}_{n}^{(2)} \right\rangle + \left\langle \mathcal{M}_{n}^{(1)} \middle| \mathcal{M}_{n}^{(1)} \right\rangle \right) \mathrm{F}_{n}$$

$\hat{\sigma}_{ab}^{\mathrm{C2}} = (\text{double convolution})\, \mathrm{F}_n$ $\qquad$ $\hat{\sigma}_{ab}^{\mathrm{C1}} = (\text{single convolution})\, \mathrm{F}_{n+1}$

Partonic cross section beyond NLO

$$\hat{\sigma}_{ab}^{(2)} = \hat{\sigma}_{ab}^{\mathrm{VV}} + \hat{\sigma}_{ab}^{\mathrm{RV}} + \hat{\sigma}_{ab}^{\mathrm{RR}} + \hat{\sigma}_{ab}^{\mathrm{C2}} + \hat{\sigma}_{ab}^{\mathrm{C1}}$$



Technically substantially more complicated!

Main bottlenecks in the real-real:

- Analytic integration is more difficult
- Many possible limits → good organization principle needed
- Eventually to most important:
  numerical efficiency & automation & flexibility

Slicing and Subtraction

Slicing

- Conceptually simple
- Recycling of lower computations
- Non-local cancellations/power-corrections
  → computationally expensive

NNLO QCD schemes

qT-slicing [Catani'07]
N-jettiness slicing [Gaunt'15/Boughezal'15]
… see for example [Buonocore'25]

Subtraction

- Conceptually more difficult
- Local subtraction → efficient
- Better numerical stability
- Choices:
  - Momentum mapping
  - Subtraction terms
  - Numerics vs. analytic

Antenna [Gehrmann'05-'08]
Colorful [DelDuca'05-'15]
Sector-improved residue subtraction [Czakon'10-'14'19]
Projection [Cacciari'15]
Nested collinear [Caola'17]
Geometric [Herzog'18]
Unsubtraction [Aguilera-Verdugo'19]
…

Top-quark pair plus jet production

Sector-improved residue subtraction [Czakon'10-'14'19]
full NNLO QCD except for two-loop virtuals:

$$F^{(2)}_{\text{l.c. resc.}}(\mu^2) = \frac{F^{(0)}}{F^{(0)}_{\text{l.c.}}} F^{(2)}_{\text{l.c.}}(Q^2) + \sum_{i=0}^{4} c_i \log^i(\mu^2/Q^2)$$

$$F^{(0)} = \langle \mathcal{F}^{(0)} | \mathcal{F}^{(0)} \rangle \,, \quad F^{(1)} = 2\,\mathrm{Re}(\langle \mathcal{F}^{(0)} | \mathcal{F}^{(1)} \rangle) \,,$$
$$F^{(2)} = 2\,\mathrm{Re}(\langle \mathcal{F}^{(0)} | \mathcal{F}^{(2)} \rangle) + \langle \mathcal{F}^{(1)} | \mathcal{F}^{(1)} \rangle \,,$$

	σ^{LO} [pb]	σ^{NLO} [pb]	σ^{NNLO} [pb]
$\mu_R = \mu_F = H_T/4$			
soft	$341.9(0.2)^{+48.0\%}_{-30.2\%}$	$383.0(1.3)^{+0.984\%}_{-9.9\%}$	$362.9(10.1)^{+0.73\%}_{-4.27\%}$
hard	$60.8(0.0)^{+49.0\%}_{-30.6\%}$	$74.1(0.2)^{+4.27\%}_{-12.3\%}$	$70.4(0.6)^{+0.593\%}_{-6.47\%}$
$\mu_R = \mu_F = H_T/2$			
soft	$238.5(0.1)^{+43.3\%}_{-28.3\%}$	$345.1(0.8)^{+11.0\%}_{-14.4\%}$	$363.0(6.2)^{+0.698\%}_{-4.73\%}$
hard	$42.2(0.0)^{+44.1\%}_{-28.6\%}$	$65.0(0.1)^{+14.1\%}_{-15.9\%}$	$70.0(0.4)^{+1.08\%}_{-6.04\%}$
$\mu_R = \mu_F = H_T$			
soft	$171.1(0.1)^{+39.4\%}_{-26.6\%}$	$295.5(0.6)^{+16.8\%}_{-16.2\%}$	$345.8(4.1)^{+5.06\%}_{-7.97\%}$
hard	$30.1(0.0)^{+40.1\%}_{-26.9\%}$	$54.7(0.1)^{+18.9\%}_{-17.3\%}$	$65.8(0.3)^{+6.43\%}_{-9.36\%}$

Leading colour amplitudes from:

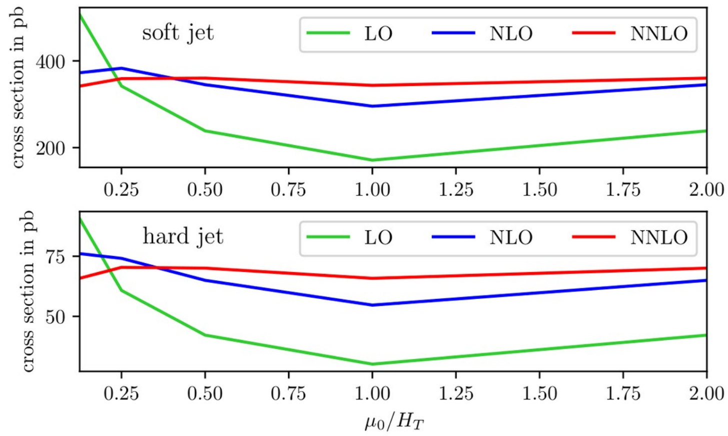
Double virtual QCD corrections to $t\bar{t}$+jet production at the LHC,
Badger, Becchetti, Brancaccio, Czakon, Hartanto, **Poncelet**, Zoia
[arxiv:2511.11424]

LHC phase spaces at 13 TeV:

$$\mu_R = \mu_F = \mu_0 = H_T/n$$
$$H_T = M_T(t) + M_T(\bar{t}) + p_T(j_1)$$

Soft-jet: $p_T(j) > 30\,\text{GeV},\ |y(j)| < 2.5$

Hard-jet: $p_T(j) \geq 120\,\text{GeV},\ p_T(j) \geq 0.4(p_T(t) + p_T(\bar{t})),\ |y(j)| < 2.4$

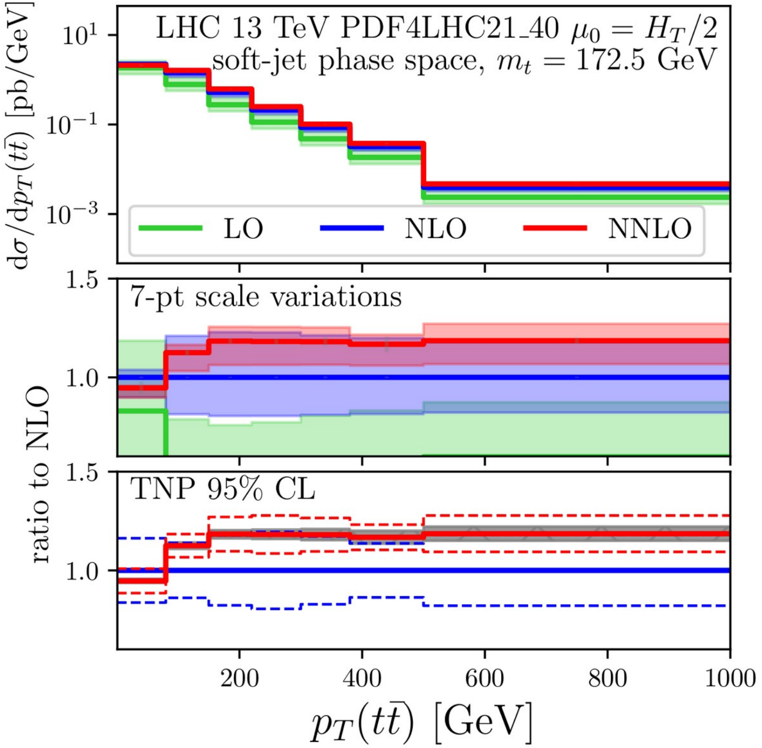


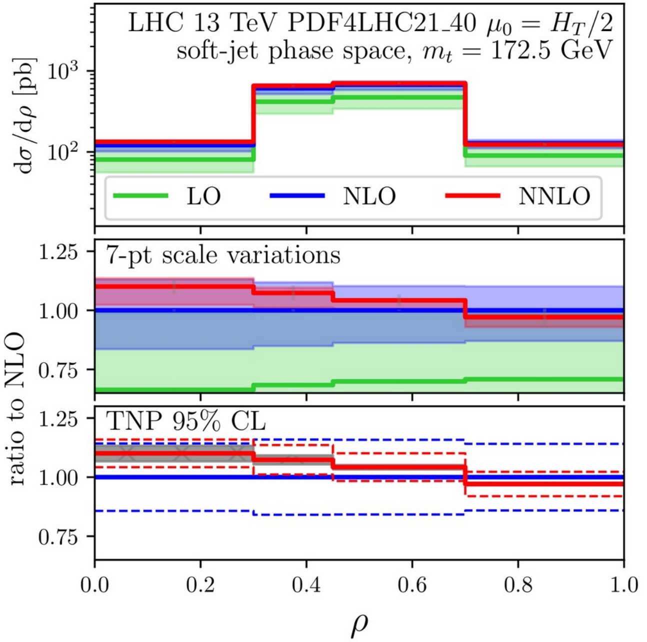
Top-quark pair+jet production: differential observables

→ typical perturbative convergence
→ impact of leading-colour:

$$\frac{F^{(2)}_{l.c.}}{F^{(0)}_{l.c.}} \to \frac{F^{(2)}_{l.c.}}{F^{(0)}_{l.c.}} + \frac{\theta}{N_c^2}\frac{F^{(2)}_{l.c.}}{F^{(0)}_{l.c.}}$$

$$\theta \in [-1, 1]$$

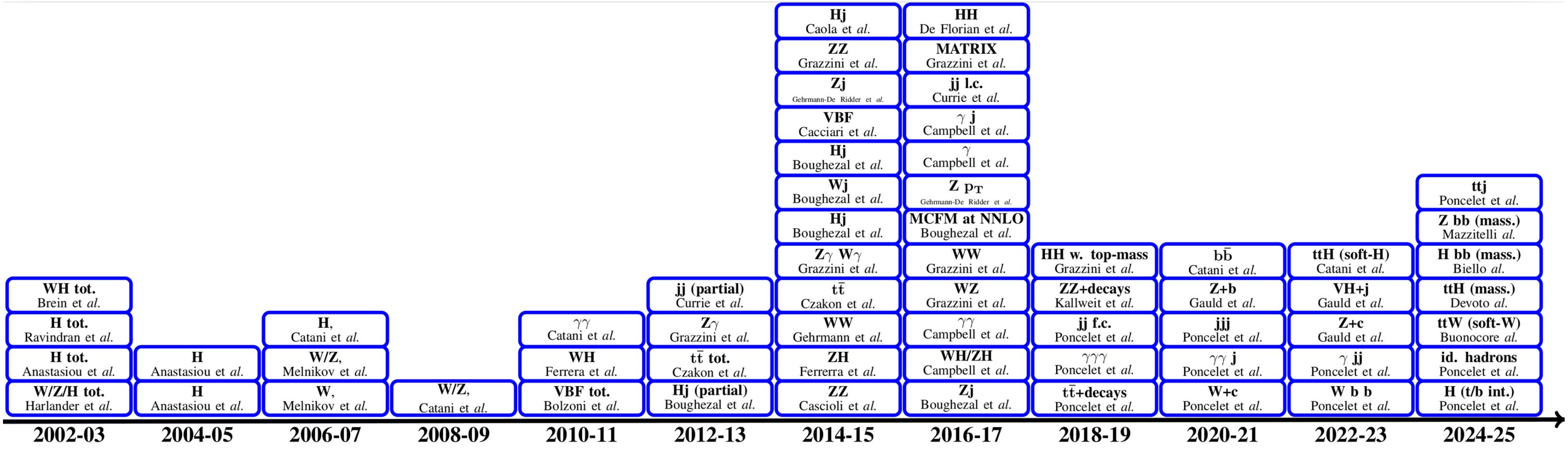




$$\rho = \frac{2m_0}{m(t\bar{t}j)}$$

Future phenomenological applications
→ precision top-quark mass extraction
→ strong coupling through ratios?
→ input for N3LO top-pair with slicing...

Overview status of NNLO QCD cross section computations

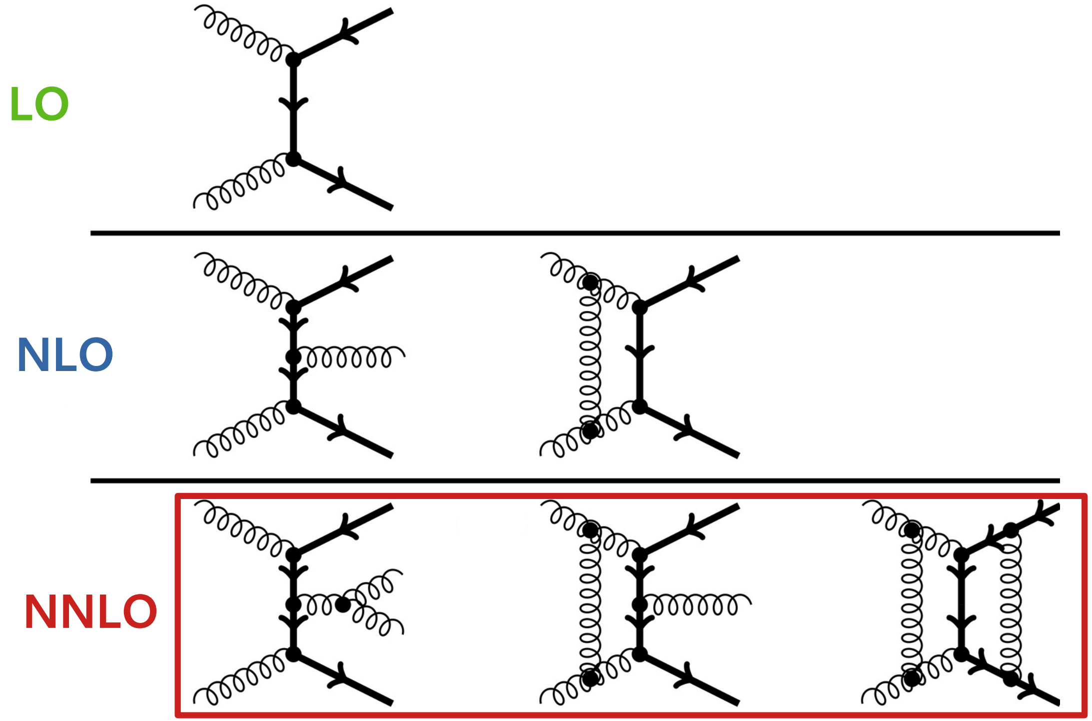


All 2 → 2 SM processes, 2 → 3+ only **limited by**

→ **the available two-loop matrix elements**

→ **numerical/computational challenges**

NNLO QCD challenges: numerics



How to deal with the **numerics**?

→ stability of loop-amplitudes?

→ two-loops in the bulk

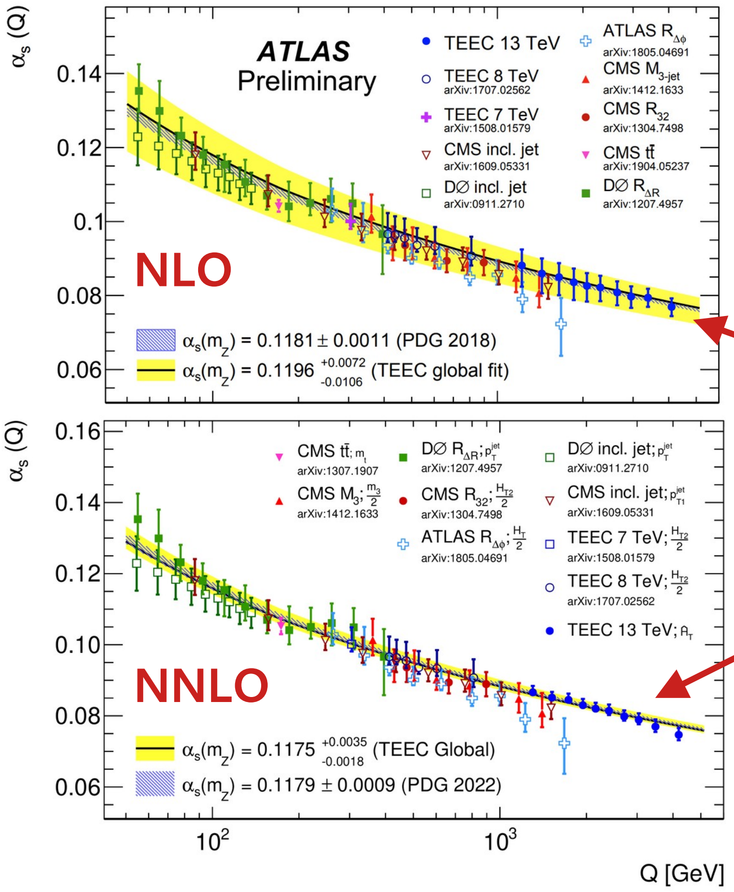
→ one-loops in the IR

→ **cross section integration**?

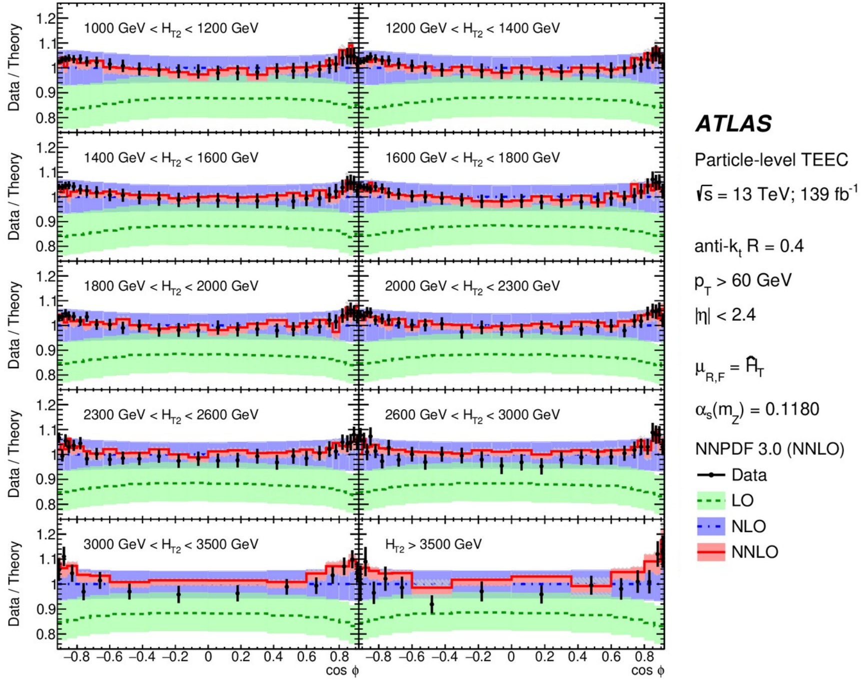
→ large integrand variance

→ negative weights

Transverse-Energy-Energy-Correlator TEEC



$$\frac{1}{\sigma_2}\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\Delta\phi} = \frac{1}{\sigma_2}\sum_{ij}\int \frac{\mathrm{d}\sigma\, x_{\perp,i}x_{\perp,j}}{\mathrm{d}x_{\perp,i}\mathrm{d}x_{\perp,j}\mathrm{d}\cos\Delta\phi_{ij}}\delta(\cos\Delta\phi - \cos\Delta\phi_{ij})\mathrm{d}x_{\perp,i}\mathrm{d}x_{\perp,j}\mathrm{d}\cos\Delta\phi_{ij}\,,$$



computationally **very costly** enterprise…
(~200MCPUh)

Monte Carlo integration

$$\sigma_{h_1 h_2 \to X} = \sum_{ij} \int_0^1 \int_0^1 \mathrm{d}x_1 \mathrm{d}x_2 \phi_{i,h_1}(x_1, \mu_F^2) \phi_{j,h_2}(x_2, \mu_F^2) \hat{\sigma}_{ij \to X}(\alpha_s(\mu_R^2), \mu_R^2, \mu_F^2)$$

- Numerical integration of highly dimensional integrands → Monte Carlo Sampling

Integral

$$I = \int_{\mathbf{x} \in \Omega} \mathrm{d}\mathbf{x} f(\mathbf{x})$$

MC estimate

$$\hat{I} = \frac{1}{N} \sum_{i=1}^{N} f(\mathbf{x}_i) \,,$$

MC error estimate

$$\delta \hat{I} = \sqrt{\frac{1}{N-1} \left(\frac{1}{N} \sum_{i=1}^{N} f^2(\mathbf{x}_i) - \hat{I}^2 \right)}$$

- Variance reduction techniques improve performance, mapping $\mathbf{H} : \Omega \to \Omega, \mathbf{x} \mapsto \mathbf{H}(\mathbf{x})$

$$I = \int_{\mathbf{H}(\mathbf{x}) \in \Omega} \mathrm{d}\mathbf{H} \frac{f(\mathbf{x})}{h(\mathbf{x})}$$

$$h(\mathbf{x}) = \left| \det \left(\frac{\partial \mathbf{H}(\mathbf{x})}{\partial \mathbf{x}} \right) \right|$$

Find with $h(\mathbf{x})$ adaptive MC techniques: VEGAS [Lepage'78], Parni [Hameren'14],
ML techniques: Normalising Flows Iflow [Bothmann'20] Madnis [Heimel'22], ...

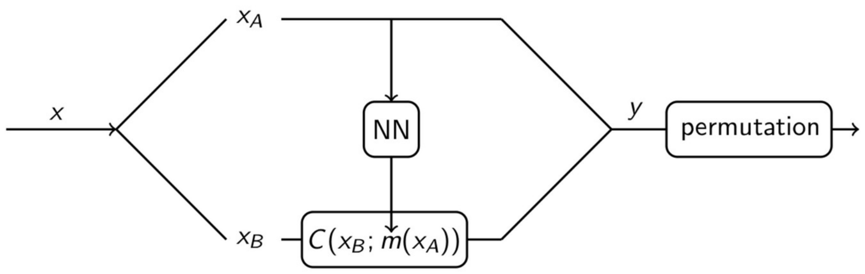
Coupling Layer Normalizing Flow

Based on the i-flow paper: 2001.05486

Series of bijections: $$\vec{x}_K = c_K(c_{K-1}(\cdots c_2(c_1(\vec{x}))))$$

Distribution: $$g_K(\vec{x}_K) = g_0(\vec{x}_0) \prod_{k=1}^{K} \left| \frac{\partial c_k(\vec{x}_{k-1})}{\partial \vec{x}_{k-1}} \right|^{-1}, \quad \text{where} \quad \begin{cases} \vec{x}_0 = \vec{x} \\ \vec{x}_k = c_k(\vec{x}_{k-1}) \end{cases}$$

Structure of a single coupling layer:

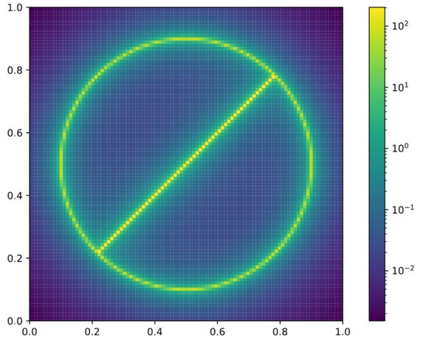


$$\begin{aligned} x'_A &= x_A, & A &\in [1, d]\ , \\ x'_B &= C(x_B; m(\vec{x}_A)), & B &\in [d+1, D]\ . \end{aligned}$$

A toy example

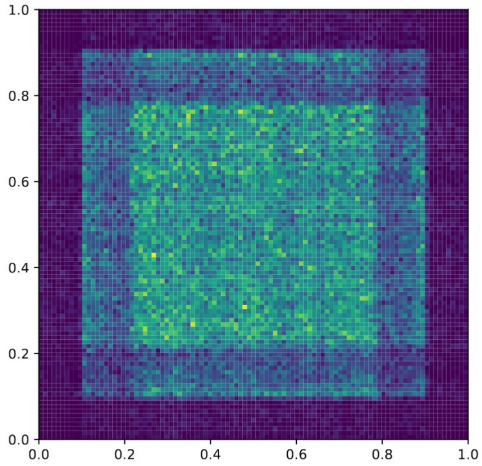
Multi-modular function ("stop-sign"):

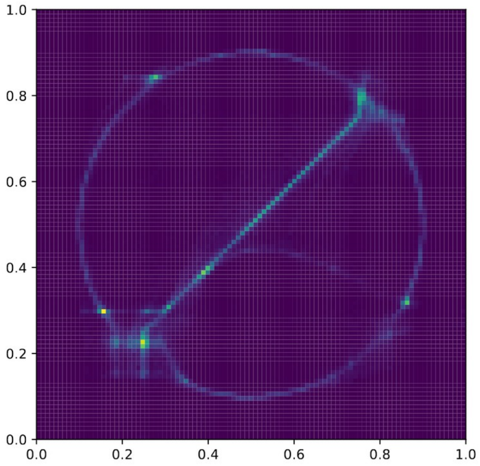
$$f(x,y) = \frac{1}{2\pi^2} \cdot \frac{\Delta r}{\left(\sqrt{(x-x_0)^2+(y-y_0)^2} - r_0\right)^2 + (\Delta r)^2} \cdot \frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2}}$$
$$+ \frac{1}{2\pi r_0} \cdot \frac{\Delta r}{((y-y_0)-(x-x_0))^2 + (\Delta r)^2} \cdot \Theta\left(r_0 - \sqrt{(x-x_0)^2+(y-y_0)^2}\right).$$



Sampling densities:

VEGAS

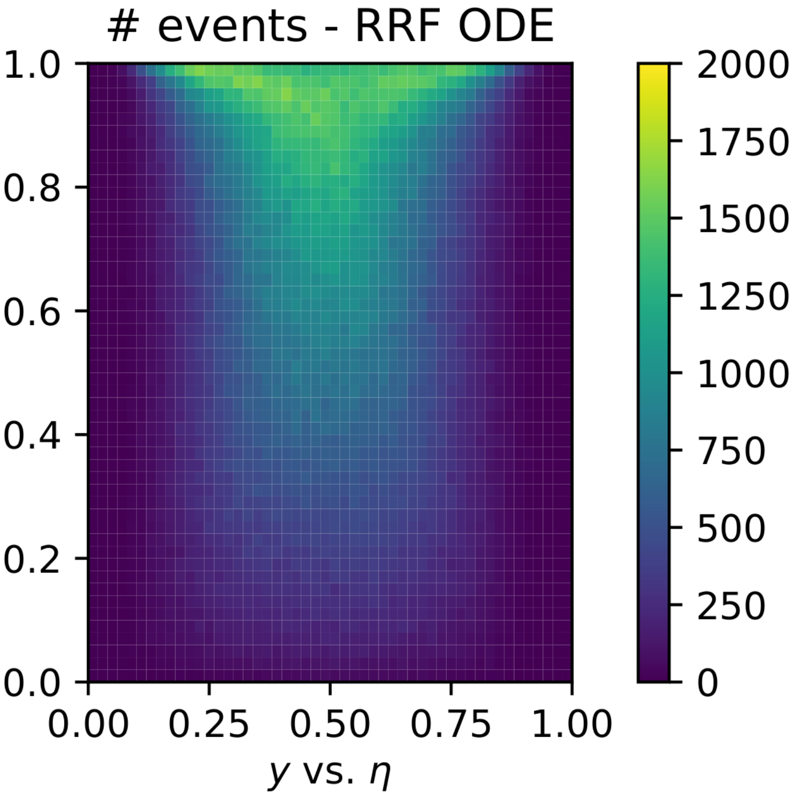


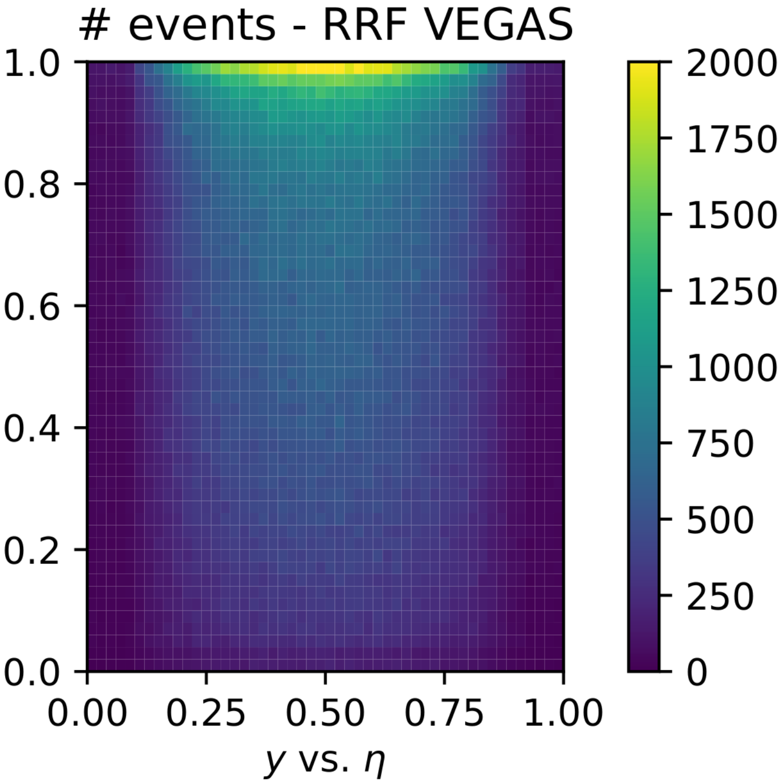


Coupling-Layer Flow

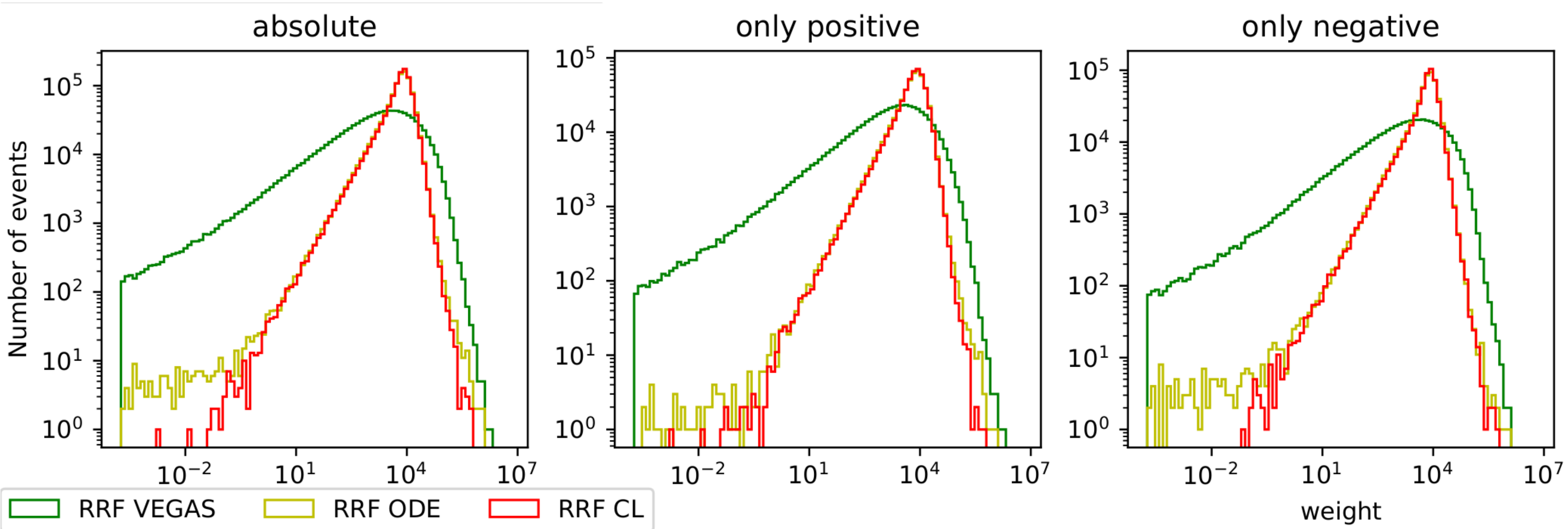
Non-factorizing phase space features

Sampling NNLO QCD phase space with normalizing flows
Janßen, **Poncelet**, Schumann JHEP 10 (2019), 262





Weight distribution for double real



Non-positive definite integrands

- Non-definite integrands introduce new challenges

  → cancellation between +/- parts increase the variance

- Consider extreme case: $|f(x)|/h(x) = w = \text{const.}$

  MC estimate:

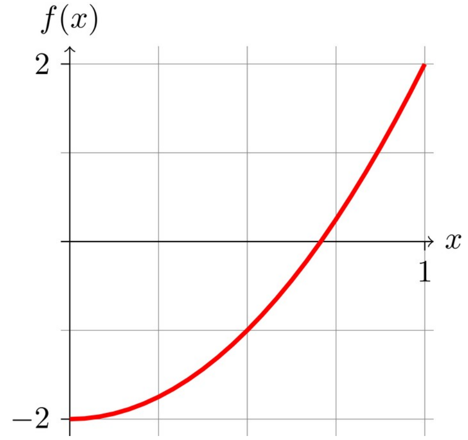
$$\hat{I} = w\frac{N_+ - N_-}{N} \equiv w(2\alpha - 1) \qquad \alpha = N_+/N$$

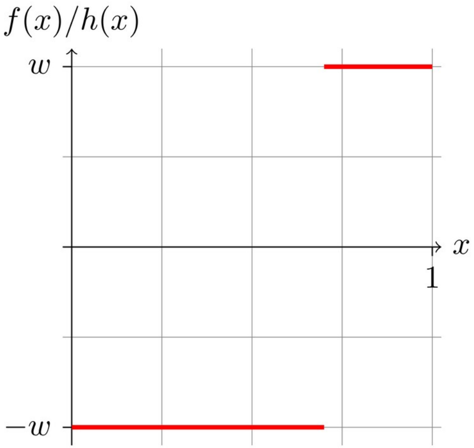
- Lower bound on variance:

$$\text{Var}(\hat{I}) = w^2 - w^2(2\alpha-1)^2 = w^2(4\alpha(1-\alpha))$$

→ relative uncertainty: $\frac{\delta\hat{I}}{\hat{I}} = \frac{1}{\sqrt{N-1}}\frac{\sqrt{\alpha(1-\alpha)}}{\alpha - \frac{1}{2}}$

Rephrased: at some point it doesn't matter any more how good your adaptive MC is...





Stratification of signed integrands

There are ways around:

1) Add a large constant

2) Stratification: $f(\mathbf{x}) = f_+(\mathbf{x}) + f_-(\mathbf{x})\,, \quad \text{with} \quad f_\pm(\mathbf{x}) = \Theta(\pm f(\mathbf{x}))\, f(\mathbf{x})$

$$\Rightarrow \quad I = \int_{\mathbf{H}_+(\mathbf{x})\in\Omega} \mathrm{d}\mathbf{H}_+ \, \frac{f_+(\mathbf{x})}{h_+(\mathbf{x})} + \int_{\mathbf{H}_-(\mathbf{x})\in\Omega} \mathrm{d}\mathbf{H}_- \, \frac{f_-(\mathbf{x})}{h_-(\mathbf{x})}$$

"two independent integrals"

$$\hat{I}_{\text{strat}} = \hat{I}_+ + \hat{I}_- = \frac{1}{N_+}\sum_{i=1}^{N_+} \frac{f_+(\mathbf{x}_i)}{h_+(\mathbf{x}_i)} + \frac{1}{N_-}\sum_{i=1}^{N_-} \frac{f_-(\mathbf{x}_i)}{h_-(\mathbf{x}_i)}$$

$$\delta\hat{I}_{\text{strat}} = \sqrt{\frac{1}{N-1}\left[\frac{N}{N_+}\mathrm{Var}(\hat{I}_+) + \frac{N}{N_-}\mathrm{Var}(\hat{I}_-)\right]}$$

$$\mathrm{Var}(\hat{I}_\pm) = \frac{1}{N_\pm}\sum_{i=1}^{N_\pm}\left(\frac{f_\pm(\mathbf{x}_i)}{h_\pm(\mathbf{x}_i)}\right)^2 - \hat{I}_\pm^2$$

+ The total variance is now bounded by the individual variances

- The mappings are more complicated (need high phase space efficiency)

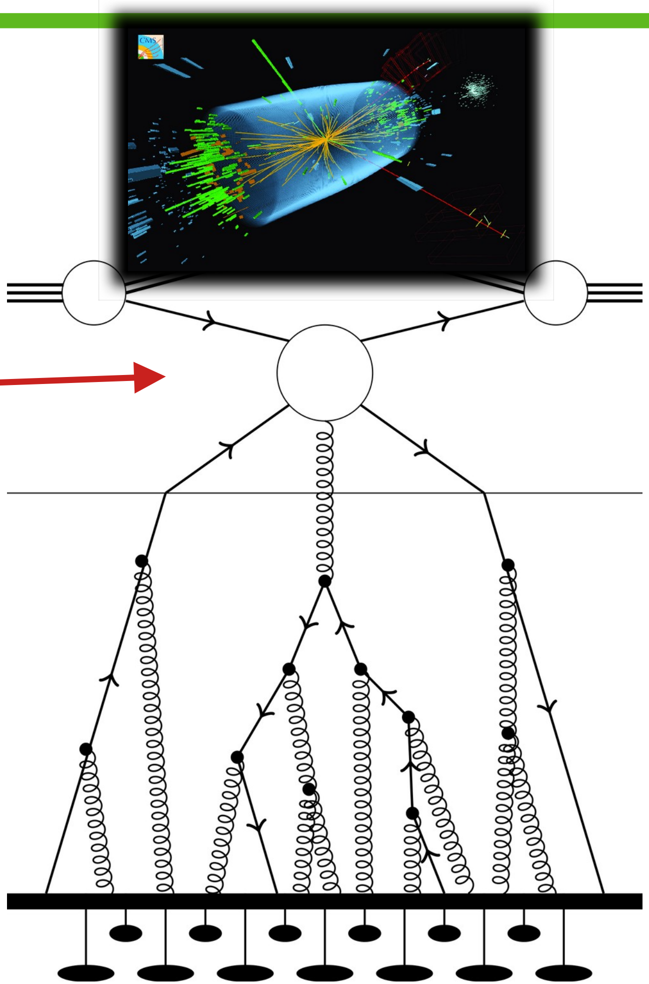
→ ideal case for **flows**!

Beyond fixed-order perturbation theory

Guiding principle: factorization
”What you see depends on the energy scale”

In Quantum Chromodynamics (QCD):

$Q \gg \Lambda_{\mathrm{QCD}}$ **Fixed-order perturbation theory**
scattering of individual partons



Beyond fixed-order perturbation theory

Guiding principle: factorization
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In Quantum Chromodynamics (QCD):

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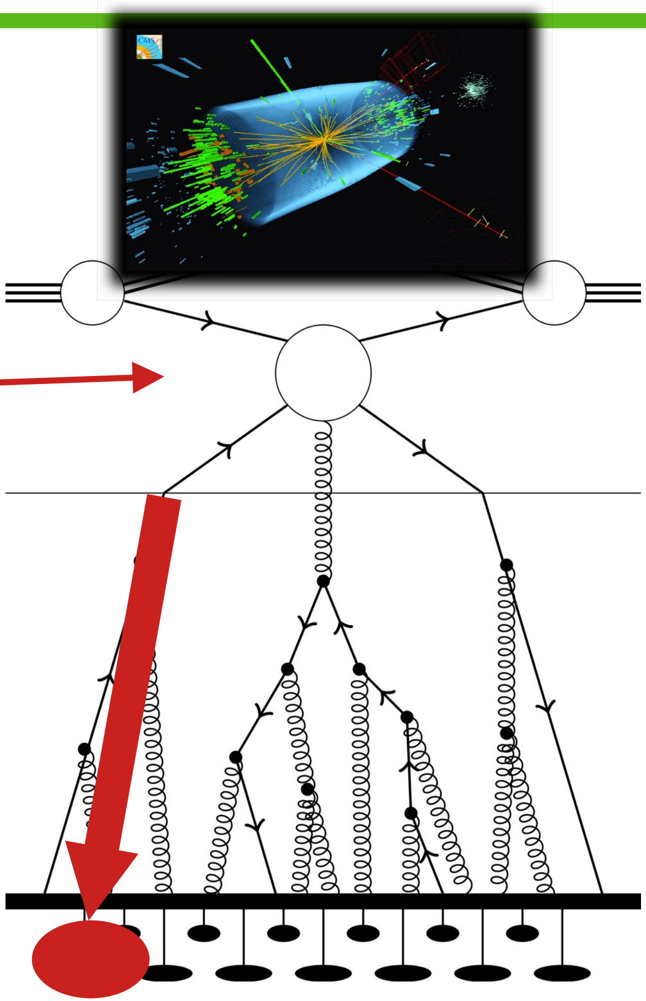
scattering of individual partons

Parton to identified object transition: "**Fragmentation**"
→ Resummation of collinear logs through 'DGLAP'
→ Non perturbative fragmentation functions (FF)
Example: B-hadrons in e+e-

$$\frac{d\sigma_B(m_b,z)}{dz} = \sum_i \left\{ \frac{d\sigma_i(\mu_{Fr},z)}{dz} \otimes D_{i\to B}(\mu_{Fr},m_b,z) \right\}(z) + \mathcal{O}(m_b^2)$$



Identified hadrons

Inclusion of fragmentation through NNLO QCD:

B-hadron production in NNLO QCD: application to LHC tt events with leptonic decays, NNLO B-fragmentation fits and their application to tt production and decay at the LHC
Czakon, Generet, Mitov, **Poncelet**
[JHEP 10(2021)216, JHEP 03(2023)251]

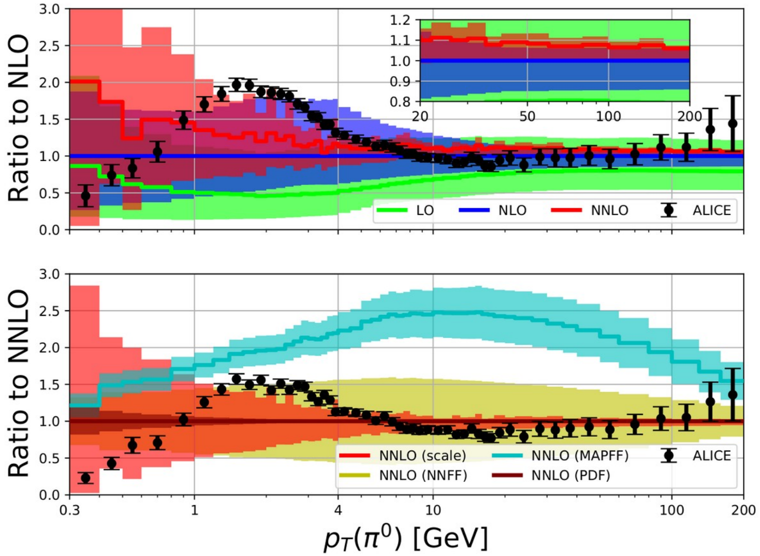
Open B-Hadron Production at Hadron Colliders in QCD at NNLO and NNLL Accuracy,
Czakon, Generet, Mitov, **Poncelet**
[PRL 135 (2025) 16]

Identified Hadron Production at Hadron Colliders in NNLO QCD,
Czakon, Generet, Mitov, **Poncelet**
[PRL 135 (2025) 17]

Associated production of a W-boson and a charm meson at NNLO in QCD,
Generet, **Poncelet,** Muškinja
[2510.24525]

Example: pion production

$$d\sigma_{pp\to h}(p) = \sum_i \int dz\; d\hat{\sigma}_{pp\to i}\left(\frac{p}{z}\right) D_{i\to h}(z)$$



→ NNLO FF fits from LHC data!

Fragmentation & jet substructure

Analogy: jet of given size&energy ⟷ massive 'hadron' $m(j) \sim R \cdot p_T$

Semi-inclusive jet function [1606.06732, 2410.01902]:

$$\frac{d\sigma_{\mathrm{LP}}}{dp_T d\eta} = \sum_{i,j,k} \boxed{\int_{x_{i,\min}}^{1} \frac{dx_i}{x_i} f_{i/P}(x_i,\mu) \int_{x_{j,\min}}^{1} \frac{dx_j}{x_j} f_{j/P}(x_j,\mu) \int_{z_{\min}}^{1} \frac{dz}{z} \mathcal{H}_{ij}^{k}(x_i, x_j, p_T/z, \eta, \mu)} \times \boxed{J_k\left(z, \ln\frac{p_T^2 R^2}{z^2\mu^2}, \mu\right)},$$

The same hard function as for identified hadrons → NNLO QCD!

Modified RGE (not quite DGLAP):

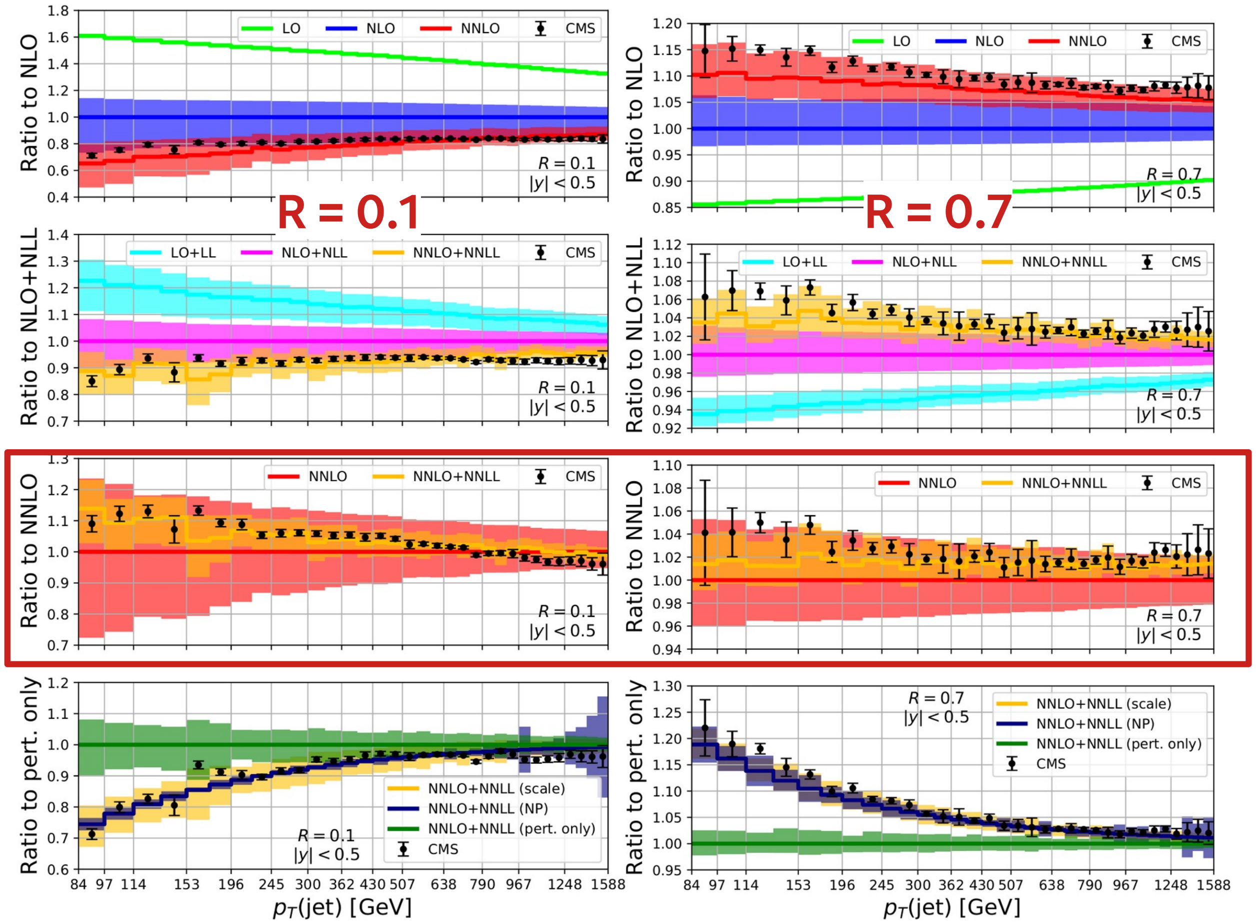
$$\frac{d\vec{J}\left(z, \ln\frac{p_T^2 R^2}{z^2\mu^2}, \mu\right)}{d\ln\mu^2} = \int_z^1 \frac{dy}{y} \vec{J}\left(\frac{z}{y}, \ln \boxed{\frac{y^2 p_T^2 R^2}{z^2\mu^2}}, \mu\right) \cdot \widehat{P}_T(y)$$

[2402.05170,2410.01902]

Jet-substructure observables like energy-energy correlators obey similar factorization!
→ **great opportunity for precision QCD phenomenology in jets at the LHC**

First step: small-R jets through NNLO+NNLL

Small radius inclusive jet production at the LHC through NNLO+NNLL,
Generet,Lee, Moult, **Poncelet,** Zhang
[JHEP 08 (2025), 015]



Absolute spectra

+

Comparison to ‘triple’ differential measurement by CMS: [2005.05159]

Substantially improved data/theory agreement:

→ **PDF fits**

→ **benchmark for PS**

Theory picture of hadron collision events

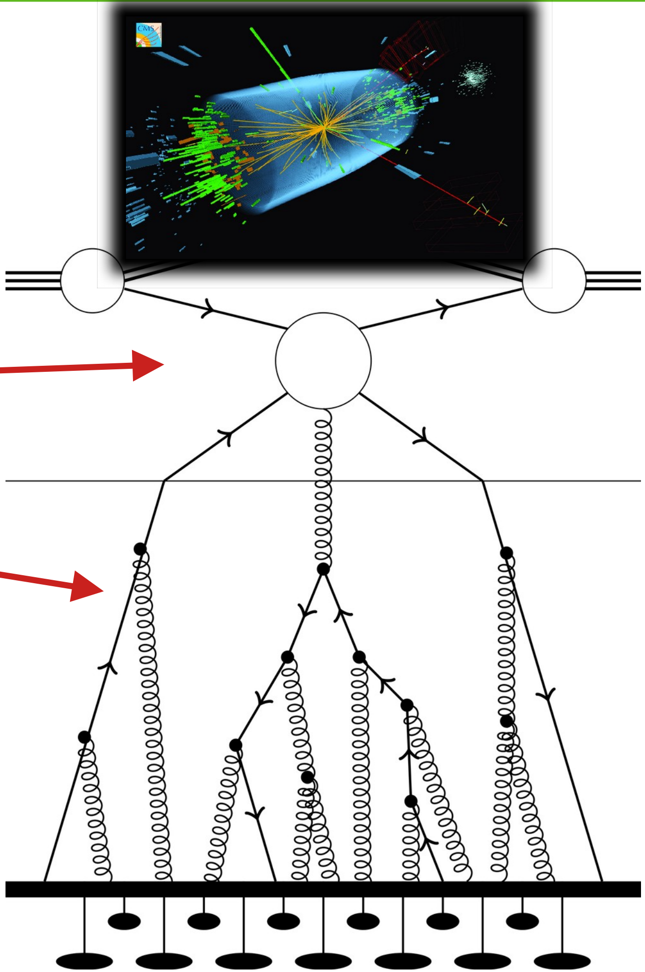
Guiding principle: factorization
"What you see depends on the energy scale"

In Quantum Chromodynamics (QCD):

$Q \gg \Lambda_{\mathrm{QCD}}$ **Fixed-order perturbation theory**
scattering of individual partons

$Q \gtrsim \Lambda_{\mathrm{QCD}}$ **Parton-shower/Resummation**
all-order bridge between perturbative and non-perturbative physics

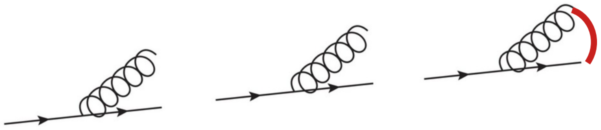
$Q \sim \Lambda_{\mathrm{QCD}}$ **"Hadronization"/MPI/...**
non-perturbative physics



Fixed-order matching to parton-showers

The challenge

Combine fixed-order with parton shower evolution
while **preserving** the precision/accuracy of both!

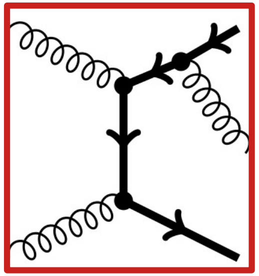


v ← shower scale

parton shower approximation: $\alpha_S^i(v)\,\ln^j(v)$
but to all orders!

accuracy: LL, NLL, NNLL, ...

“Hard” scattering scale Q



Fixed-order: expansion in $\alpha_S(Q)$

→ cross section ~ $\underbrace{e^{\sum_{ij}\alpha_S^i(v)\,\ln^j(v)} \times (\text{LO} + \text{NLO} + \text{NNLO} + \dots)}$

A matching scheme needs to avoid double counting
of the logarithmic contributions!

Matching parton showers

At NLO QCD a solved problem → a breakthrough for LHC phenomenology

Local matching NLO+PS: MC@NLO, Powheg, Nagy-Soper, …

(core of event generators Madgraph_aMC@NLO, Sherpa, Powheg+Pythia, Herwig)

>80% of all exp. LHC papers cite at least one these!

Core idea: using subtractions schemes to construct showers & matching

(subtraction terms ⟷ parton shower kernels)

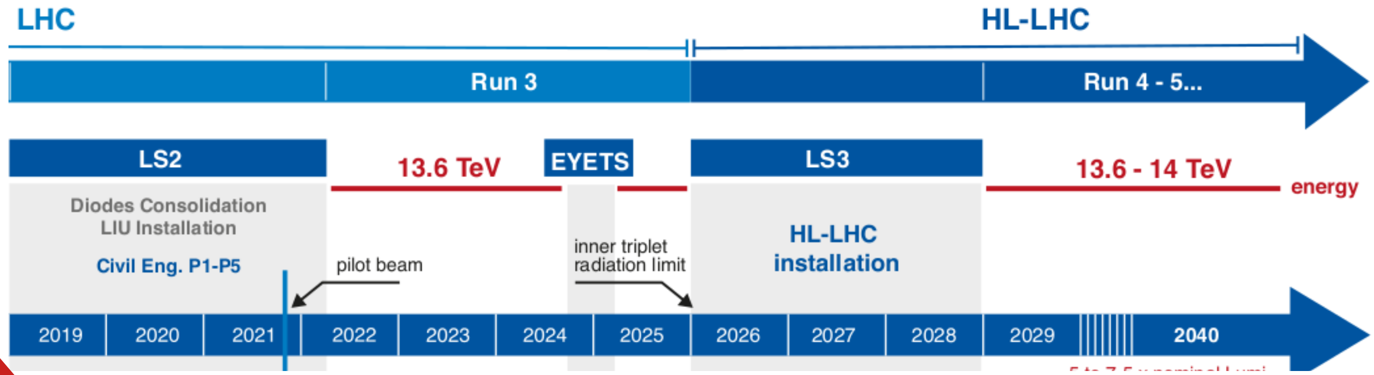
This is the **big challenge at NNLO QCD** for the theory community!

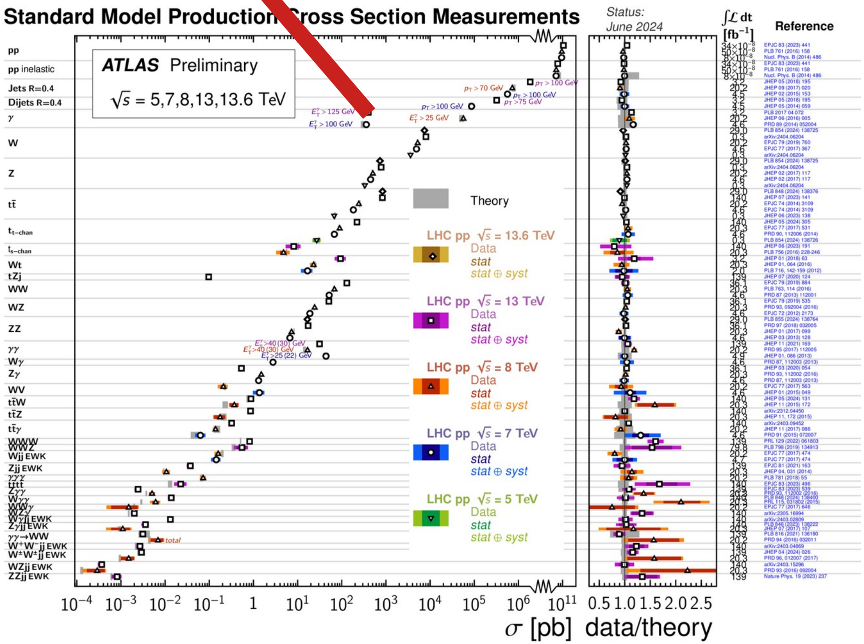
Some NNLO+PS matching approaches appeared recently but are either

- non-local → resummation/slicing based (for example: MiNNLOPS, Geneva)
  → limited generality
- or work so far only for 'simple' cases like e+e- → jets (for example: Vincia)
  → where NNLO is known analytically

No scheme so far is based on a general local subtraction.

Run 2





detectors will rebuild &
new techniques implemented

→ HL-LHC is basically a new collider
(allowing for unprecedented precision in pp)

→ let's bring theory in shape
to be able to interpret it!

???

Thank you!