CERN-TH/95-38

Higher-Order Radiative Corrections to Low-Angle Bhabha Scattering: The YFS Monte Carlo Approach§

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Abstract

We calculate the QED-corrected integrated cross-section for small-angle Bhabha scattering with kinematical cuts identical to the real experimental cuts at the LEP/SLC experiments: no distinction between electrons and photons is made. We present a new second-order Monte Carlo calculation with the exponentiation of the Yennie-Frautschi-Suura type. It provides the integrated cross-section for the low-angle Bhabha scattering process ($\vartheta < 0.060$), necessary for the luminosity measurement, with an overall precision of 0.16%. The corresponding computer program BHLUMI 4.01 is in the form of a stand-alone Monte Carlo event generator. The complete and explicit definition of the multiphoton matrix element is given. Examples of numerical results from Monte Carlo phase-space integration are demonstrated and discussed. The analytical integration of the cross section for simplified cuts, keeping terms up to third order, used for cross-checks of the Monte Carlo calculation, is also presented.

To appear in Physics Letters

§ Work supported in part by Polish Government grant KBN 2P30225206 and by the US DoE contract DE-FG05-91ER40627.

CERN-TH/95-38
February 1995

The luminosity at LEP/SLC is measured with the help of the low-angle Bhabha (LABH) process, $e^+e^- \to e^+e^-$, in the angular range below 100 mrad. The luminosity determined in this way provides the absolute normalization of the cross section of all other processes in the e^+e^- scattering. The LABH cross section is therefore not generally of physical interest in itself. On the contrary, it is regarded as completely known from theory, from Quantum Electrodynamics (QED). However, although in principle the LABH cross section is calculable in perturbative QED with arbitrary precision (except for a small hadronic correction and a small Z^0 interference effect), it is subject to theoretical uncertainties due to a truncation of the perturbative expansion and to a limitation of the calculational tools (computer programs). All LEP/SLC experiments use theoretical calculations for LABH, based on works published by the present authors three years ago [1, 2]. This calculation has an overall theoretical/technical precision of 0.25% and is embodied in the form of the Monte Carlo event generator [3] BHLUMI version 2.01. This error was acceptable in 1991 but now, with an improvement in the experimental precision of a factor of ~ 4, it dominates the present overall luminosity error. It is therefore quite urgent to reduce the theoretical error of the QED calculation to a precision level of at least 0.1%. In this paper we describe a significant step in this direction, we present a new calculation with a conservative overall theoretical precision of 0.16%, which is implemented in version 4.01 of BHLUMI. In the following we shall (a) describe the new matrix element in BHLUMI 4.01, (b) present an analytical cross check which gives us confidence that it was implemented correctly, (c) present numerical results relevant for new error analysis and, finally, (d) present a discussion of the new theoretical error.

Although our aim is to define full differential distributions for the exponentiated case, it is, however, easier and more transparent to define in the first place all differential distributions for the case without exponentiation and then write down the exponentiated distributions using the previously defined distributions as building blocks. Exclusive exponentiation will be done according to the Yennie-Frautschi-Suura (YFS) method [4]. As was pointed out on many occasions [1, 2, 5] the leading logarithmic (LL) terms proportional to powers of the "big logarithm" $L = 2(\alpha/\pi)\ln(|t|/m_e^2)$ are dominant and subleading higher-order terms are often completely negligible. (Here, t is the squared four-momentum transfer as usual.) In the present work we shall reproduce correctly in the matrix element the real and virtual corrections which after phase-space integration lead to contributions of $\mathcal{O}(\alpha^1 L^1, \alpha^1 L^0, \alpha^2 L^2)$. We refer to this class of corrections as the $\mathcal{O}(\alpha^2)_{\rm prag}$ approximation. Terms of $\mathcal{O}(\alpha^2 L^1)$ (in the above meaning) will be incomplete and those of $\mathcal{O}(\alpha^3 L^3)$ will be absent in the case without YFS exponentiation while most of them will be generated by the YFS exponentiation, see Ref. [5] for more discussion. (The YFS exponentiation also improves terms of $\mathcal{O}(\alpha^2 L^1)$.) We would like to strongly stress that LL approximations will be made only on the infrared finite part of the matrix element, i.e. the correct soft limit (no LL approximation!) will be kept throughout the calculation at the level of the exclusive cross sections, contrary to *naïve* exponentiation, where the correct soft photon behaviour is corrected by hand at the level of exclusive distributions[1]. The total cross

[1]See Ref. [6] for the discussion on the differences between inclusive and exclusive exponentiation.

section for $e^+(p_1)+e^-(q_1)\to e^+(p_2)+e^-(q_2)+\gamma(k_1)+\dots\ \gamma(k_n)+\gamma(k'_1)+\dots\ \gamma(k'_{n'})$ (the case without exponentiation) reads:

$$\begin{aligned}\sigma^{(r)} = \sum_{0\le n+n'\le r} \frac{1}{n!}\frac{1}{n'!} \int \frac{d^3p_2}{p_2^0} \int \frac{d^3q_2}{q_2^0} \prod_{j=1}^{n} \int_{k_j\notin\Omega_U} \frac{d^3k_j}{k_j^0} \prod_{l=1}^{n'} \int_{k'_l\notin\Omega_L} \frac{d^3k'_l}{k'^0_l}\\ \delta^{(4)}\left(p_1-p_2+q_1-q_2-\sum_{j=1}^{n}k_j-\sum_{l=1}^{n'}k'_l\right)\ D^{(r)}_{[n,n']}(k_1...k_n;k'_1...k'_{n'}),\end{aligned} \tag{1}$$

where $\Omega_{U,L}$ are 3-dimensional regions surrounding the infrared soft singularity point which is excluded in this way from the multiphoton phase space. Usually this is done by requiring that the photon energy be above some value $E_{\max}$ in a certain reference frame[2]. Virtual contributions and real soft photon contributions below $E_{\max}$ (regularized typically with a photon mass λ) are combined and included in the corresponding $D^{(r)}_{[n,n']}$.

The $\mathcal{O}(\alpha^r)_{\rm prag}$, $r=0,1,2$ differential distribution with zero real photons reads:

$$\begin{aligned} D^{(r)}_{[0,0]} &= \frac{4\pi\alpha^2}{t_p t_q}\, b_0\,(1+v^{(r)}),\quad v^{(0)}=0,\quad v^{(1)}=2\gamma\ln\Delta+\frac{3}{2}\gamma-\frac{\alpha}{\pi},\\ v^{(2)} &= 2\gamma\ln\Delta+\frac{1}{2}\gamma^2\ln^2\Delta+(1+2\gamma\ln\Delta)\left(\frac{3}{2}\gamma-\frac{\alpha}{\pi}\right)+\frac{9}{8}\gamma^2,\\ \gamma &= 2\frac{\alpha}{\pi}\left(\ln\frac{1}{\delta}-1\right),\quad \delta=\frac{m_e^2}{|t|},\quad t=(p_1-p_2-\sum_i k_i)^2,\\ b_0 &= \frac{1}{2}(1+(1-\zeta)^2),\quad \zeta=\frac{|t|}{s}.\end{aligned} \tag{2}$$

The case of $r=0$ with $v^{(0)}=0$ represents the Born approximation, $v^{(1)}$ represents the exact $\mathcal{O}(\alpha^1)$ from the Feynman rules (up-down interference excluded, see Ref. [7]), while $9\gamma^2/8$ in $v^{(2)}$ is the second-order leading logarithmic non-infrared correction. At this point we should define the infrared domains $\Omega_{U,L}$ and the infrared cut parameter Δ. Let us postpone their definition to the moment after we define $D^{(r)B}_{[1,0]}$. The $\mathcal{O}(\alpha)_{\rm prag}$ and $\mathcal{O}(\alpha^2)_{\rm prag}$ single real photon emission distributions read as follows

$$\begin{aligned} D^{(r)}_{[1,0]}(k_1) =& \frac{4\pi\alpha^2}{t_p t_q}\ \tilde S_p(\tilde\alpha_1,\tilde\beta_1)\left(1+v^{(r)}_{[1,0]}(\tilde\alpha_1,\tilde\beta_1)\right)H(\tilde\alpha_1,\tilde\beta_1,\psi_p),\\ H(\tilde\alpha_1,\tilde\beta_1,\psi_p) =& (1+\delta^*_p(\tilde\alpha_1,\tilde\beta_1))\\ & \frac{1}{4}\left[1+\left(1-\frac{\zeta}{1-\tilde\alpha_1}\right)^2+R(\tilde\alpha_1,\tilde\beta_1,\psi_p)^2+(R(\tilde\alpha_1,\tilde\beta_1,\psi_p)-(1-\tilde\beta_1)\zeta)^2\right],\\ R(\tilde\alpha_1,\tilde\beta_1,\psi_p) =& (1-\tilde\alpha_1)(1-\tilde\beta_1)+2\cos\psi_p\sqrt{\tilde\alpha_1\tilde\beta_1(1-\tilde\alpha_1)(1-\tilde\beta_1)}+\tilde\alpha_1\tilde\beta_1,\end{aligned} \tag{3}$$

[2]The reference frame might be different for upper and lower lines.

where

$$v^{(1)}_{[1,0]} = 0, \quad v^{(2)}_{[1,0]} = (\gamma_p + \gamma_q)\ln\Delta + \frac{3}{2}\gamma - \frac{\alpha}{\pi} - \frac{1}{2}\gamma\ln(1-\tilde{\alpha}_1-\tilde{\beta}_1+\tilde{\alpha}_1\tilde{\beta}_1)$$
$$\delta^*_p(\tilde{\alpha}_1,\tilde{\beta}_1) = \delta_p\frac{1}{\tilde{\alpha}_1\tilde{\beta}_1}\,\frac{(\tilde{\alpha}_1^2+\tilde{\beta}_1^2)^2}{(1+(1-\tilde{\alpha}_1)^2(1-\tilde{\beta}_1)^2)}$$
$$\gamma_p = 2\frac{\alpha}{\pi}\left(\ln\frac{1}{\delta_p}-1\right),\ \gamma_q = 2\frac{\alpha}{\pi}\left(\ln\frac{1}{\delta_q}-1\right),$$
$$\delta_p = m_e^2/|t_p|,\ \delta_q = m_e^2/|t_q|,\ t_p=(p_1-p_2)^2,\ t_q=(q_1-q_2)^2,$$

and

$$\tilde{S}_p(k_1) = \frac{\alpha}{4\pi^2}\left(\frac{2p_1p_2}{(k_1p_1)(k_1p_2)} - \frac{m_e^2}{(k_1p_1)^2} - \frac{m_e^2}{(k_1p_2)^2}\right) \tag{4}$$

is the standard YFS soft factor. The lower line distribution $D^{(r)}_{[0,1]}(k_1)$ is defined in an analogous way[3]. The first-order distribution ($r=1$) has no virtual corrections (tree level), the second-order distribution ($r=2$) includes the one-loop virtual photon. The above distributions are given in terms of the normalized Sudakov variables $\tilde{\alpha}_i$ and $\tilde{\beta}_i$ defined exactly as in paper [3]. The angle[4] ψ_p (present implicitly in the formulas of Ref. [3]) is defined in the frame QMS_p where $Q = p_2 - p_1 = (0,0,0,|t|)$ and $p_1 = (E_1,0,0,-|p_1|)$. It is an angle between two planes, one spanned with $\vec{Q}$ and $\vec{p}_2$ and another spanned with $\vec{Q}$ and $\vec{q}_1$. In eq. (3) all distributions except $v^{(2)}_{[1,0]}$ originate from the tree level $\mathcal{O}(\alpha^1)$ Feynman diagrams. The non-infrared one-loop $\mathcal{O}(\alpha^2)$ virtual correction in the $v^{(2)}_{[1,0]}$ comes from the LL ansatz. Let us stress again that in the $v^{(2)}_{[1,0]}$ the infrared term proportional to $\ln\Delta$ is totally constrained by the YFS soft limit derived in Ref. [4] (from the Feynman rules). The non-infrared terms in $v^{(2)}_{[1,0]}$ are obtained from a double convolution of the non-singlet Altarelli-Lipatov splitting function[5].

Now, having defined our kinematics, we define the infrared domains

$$\{\Omega_U : \max(\tilde{\alpha}_i,\tilde{\beta}_i) < \Delta\}, \qquad \{\Omega_L : \max(\tilde{\alpha}'_j,\tilde{\beta}'_j) < \Delta\}, \tag{5}$$

which enter in many places throughout our calculation. Needless to say, nothing depends on the actual choice of $\Omega_{U,L}$ and we always witness perfect cancellations of the infrared real and virtual divergences.

The pure $\mathcal{O}(\alpha^2)$ distribution for double photon emission from the upper line is based on the LL ansatz and the proper (YFS) soft limit. The main ingredient in the ansatz

[3] At first sight the above formulas look different from those in Ref. [3], but in fact, in $\mathcal{O}(\alpha)$ (no exponentiation), they are completely equivalent.

[4] The angle ψ_p was also defined and used explicitly in Ref. [7].

[5] The residual freedom in the definition of the non-infrared terms in $v^{(2)}_{[1,0]}$ was fixed in such a way that the latter expression for the virtual $\mathcal{O}(\alpha^2)$ correction to $\bar{\beta}_1$ is simpler.

is the double convolution of the non-singlet splitting LL kernel for the longitudinal momenta. Both photons feature non-collinear transverse momentum distributions. It is very important that we require the distribution to have the *correct soft limit* both in the case when two photons are soft and in the case when one of them is soft and the other one is hard. This is fulfilled by taking the product of the $\mathcal{O}(\alpha)$ single photon distribution for the harder photon and the soft factor times the splitting kernel for the softer one. The advantage of the LL ansatz (as compared to exact expressions known in the literature) is that it is quick in the computer evaluation and its LL content is manifest. There is obviously a freedom in the choice of the above LL ansatz (up to non-infrared $\mathcal{O}(\alpha^2 L)$ terms). It was exploited in such a way that the analytical integration (see later) over the phase space is feasible and as simple as possible. The complete ansatz reads as follows:

$$
\begin{aligned}
D^{(2)}_{[2,0]}(\tilde\alpha_1,\tilde\beta_1,\phi_1,\tilde\alpha_2,\tilde\beta_2,\phi_2) = \frac{4\pi\alpha^2}{t_p t_q} \times \\
\Big[\tilde S_p(\tilde\alpha_1,\tilde\beta_1)\tilde S_p(\tilde\alpha_2,\tilde\beta_2)\theta((\tilde\alpha_1-\tilde\beta_1)(\tilde\alpha_2-\tilde\beta_2)) \\
\theta(v_1-v_2)\,\frac{1}{2}\,\big[H(\tilde\alpha_1,\tilde\beta_1,\psi_p)\chi(v_2^*)+H(\tilde\alpha_1^*,\tilde\beta_1^*,\psi_p)\chi(v_2)\big] \\
+\tilde S_p(\tilde\alpha_1,\tilde\beta_1)\tilde S_p(\tilde\alpha_2,\tilde\beta_2)\theta(\tilde\alpha_1-\tilde\beta_1)\theta(-\tilde\alpha_2+\tilde\beta_2) \\
\theta(v_1-v_2)H(\tilde\alpha_1^*,\tilde\beta_1^*,\psi_p)\chi(v_2)+\theta(v_2-v_1)H(\tilde\alpha_2,\tilde\beta_2,\psi_p)\chi(v_1^*)+(1\rightleftarrows 2)\Big]
\end{aligned}
\tag{6}
$$

where $\chi(x)=\frac{1}{2}(1+(1-x)^2)$ and $v_i=\tilde\alpha_i+\tilde\beta_i-\tilde\alpha_i\tilde\beta_i$. Because of the "cascade" character of the double emission, we use the "starred" variables

$$
\begin{aligned}
v_1^* &= v_1/(1-v_2), & \tilde\alpha_1^* &= \tilde\alpha_1/(1-v_2), & \tilde\beta_1^* &= \tilde\beta_1/(1-v_2),\\
v_2^* &= v_2/(1-v_1), & \tilde\alpha_2^* &= \tilde\alpha_1/(1-v_1), & \tilde\beta_2^* &= \tilde\beta_1/(1-v_1).
\end{aligned}
$$

The LL ansatz (with the correct soft limit) for the simultaneous emission of one photon from the upper line and one photon from the lower line looks even simpler:

$$
\begin{aligned}
D^{(2)}_{[1,1]}(\tilde\alpha_1,\tilde\beta_1,\phi_1,\tilde\alpha_1',\tilde\beta_1',\phi_1) = &\frac{4\pi\alpha^2}{t_p t_q}\Big[\theta(v_1-v_1')\tilde S(\tilde\alpha_1,\tilde\beta_1)\tilde S(\tilde\alpha_1',\tilde\beta_1')H(\tilde\alpha_1,\tilde\beta_1,\psi_p)\chi(v_1') \\
&+\theta(v_1'-v_1)\tilde S(\tilde\alpha_1,\tilde\beta_1)\tilde S(\tilde\alpha_1',\tilde\beta_1')H(\tilde\alpha_1',\tilde\beta_1',\psi_p)\chi(v_1)\Big]
\end{aligned}
\tag{7}
$$

because we do not need to use starred variables.

The exponentiated total cross cross section for $e^+(p_1)+e^-(q_1)\to e^+(p_2)+e^-(q_2)+$

$\gamma(k_1)+\dots\ \gamma(k_n)+\gamma(k'_1)+\dots\ \gamma(k'_{n'})$ reads as follows:

$$\sigma^{(r)}=\sum_{n=0}^{\infty}\sum_{n'=0}^{\infty}\frac{1}{n!}\frac{1}{n'!}\int\frac{d^3p_2}{p_2^0}\int\frac{d^3q_2}{q_2^0}\prod_{j=1}^{n}\int_{k_j\notin\Omega_U}\frac{d^3k_j}{k_j^0}\tilde{S}_p(k_j)\prod_{l=1}^{n'}\int_{k'_l\notin\Omega_L}\frac{d^3k'_l}{k'^0_l}\tilde{S}_q(k'_l)$$
$$\delta^{(4)}\left(p_1-p_2+q_1-q_2-\sum_{j=1}^{n}k_j-\sum_{l=1}^{n'}k'_l\right)\ e^{Y_p(\Omega_U)+Y_q(\Omega_L)}$$
$$\left\{\bar{\beta}_0^{(r)}+\sum_{j=1}^{n}\frac{\bar{\beta}_{1U}^{(r)}(k_j)}{\tilde{S}_p(k_j)}+\sum_{l=1}^{n'}\frac{\bar{\beta}_{1L}^{(r)}(k'_l)}{\tilde{S}_q(k'_l)}+\sum_{n\geq j>k\geq 1}\frac{\bar{\beta}_{2UU}^{(r)}(k_j,k_k)}{\tilde{S}_p(k_j)\tilde{S}_p(k_k)}\right.$$
$$\left.+\sum_{n'\geq l>m\geq 1}\frac{\bar{\beta}_{2LL}^{(r)}(k_l,k_m)}{\tilde{S}_q(k'_l)\tilde{S}_q(k'_m)}+\sum_{j=1}^{n}\sum_{l=1}^{n'}\frac{\bar{\beta}_{2UL}^{(r)}(k_j,k'_l)}{\tilde{S}_p(k_j)\tilde{S}_q(k'_l)}\right\}. \tag{8}$$

where the YFS formfactor is defined as in Ref. [2] and the various components of the differential distribution $\bar{\beta}_0^{(r)}, r=0,1,2$, are defined according to the usual YFS scheme in terms of the $D^{(r)}$ distributions:

$$\bar{\beta}_0^{(r)}=\left\{D_{[0,0]}^{(r)}\exp(-Y_p(\Omega_U)-Y_q(\Omega_L)\right\}\Bigg|_{\mathcal{O}(\alpha^r)}, \tag{9}$$

$$\bar{\beta}_{1U}^{(r)}(k_i)=\left\{D_{[1,0]}^{(r)}(k_i)\exp(-Y_p(\Omega_U)-Y_q(\Omega_L)\right\}\Bigg|_{\mathcal{O}(\alpha^r)}-\tilde{S}_p(k_i)\,\bar{\beta}_0^{(r-1)}, \tag{10}$$

$$\bar{\beta}_{2UU}^{(2)}(k_i,k_j)=D_{[2,0]}^{(2)}(k_i,k_j)-\bar{\beta}_{1U}^{(1)}(k_i)\tilde{S}_p(k_j)-\bar{\beta}_{1U}^{(1)}(k_j)\tilde{S}_p(k_i)-\bar{\beta}_0^{(0)}\tilde{S}_p(k_i)\tilde{S}_p(k_j), \tag{11}$$

$$\bar{\beta}_{2UL}^{(2)}(k_i,k'_j)=D_{[1,1]}^{(2)}(k_i,k'_j)-\bar{\beta}_{1U}^{(1)}(k_i)\tilde{S}_q(k'_j)-\bar{\beta}_{1L}^{(1)}(k'_j)\tilde{S}_p(k_i)-\bar{\beta}_0^{(0)}\tilde{S}_p(k_i)\tilde{S}_q(k'_j). \tag{12}$$

The above $\mathcal{O}(\alpha^2)_{\rm prag}$ matrix element, with and without exponentiation, has been implemented in the Monte Carlo event generator[6] BHLUMI 4.01 [8]. The above implementation was done very carefully, i.e. the contribution from each of the ten components $\bar{\beta}_{AB}^{(r)}$ to the integrated cross section was individually checked to agree with the result of the special (semi-)analytical calculation to better than 0.03%! Let us describe briefly this very important cross-check.

Analytical integration over the multiphoton phase space for the true experimental trigger, i.e. set of cuts, is impossible. We have, therefore, defined a set of "academic" cuts, called for short an "academic trigger", for which this task is feasible and the result is not overwhelmingly complicated. We define the cuts of our "academic trigger" as follows: $|t_{\min}|<|t|<|t_{\max}|$ and $V<V_{\max}$, where the variable V represents some kind of measure of the total energy carried away by all emitted real photons. The requirement of $0<V<1$ represents the condition of completeness of the phase space and the particular

[6]In BHLUMI 4.01 the basic Monte Carlo algorithm is essentially the same as in the older version 2.01 [3].

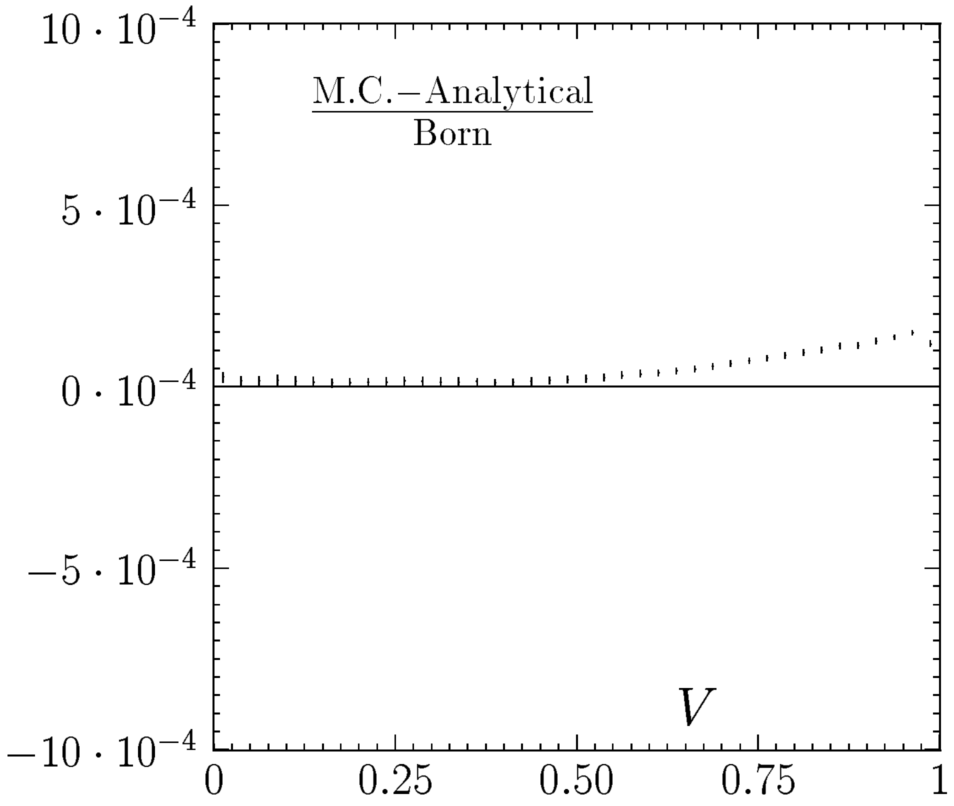


Figure 1: For a simple "academic trigger", $|t_1| < |t| < |t_2|$, we plot the difference of MC and semi-analytical results divided by Born. We have taken the limits $t_{1,2}$ representing a very narrow $t-$interval corresponding, at the Born level kinematics, to $\theta = 2.7^o$.

case $0 < V < \epsilon$ represents the condition that all photons be soft. The V-variable we actually define as:

$$V = 1 - \frac{(p_1p_2)\ |t|}{\Big[(p_1p_2)\ + \sum_j (p_1k_j)\Big]^2} \frac{(q_1q_2)\ |t|}{\Big[(q_1q_2)\ + \sum_l (q_1k'_l)\Big]^2}, \tag{13}$$

$$V = 1 - \Big(1 - \sum_j (\tilde{\alpha}_j + \tilde{\beta}_j - \tilde{\alpha}_j\tilde{\beta}_j)\Big) \Big(1 - \sum_l (\tilde{\alpha}'_l + \tilde{\beta}'_l - \tilde{\alpha}'_l\tilde{\beta}'_l)\Big). \tag{14}$$

The above two definitions are equivalent within the LL and next-to-LL approximations. With the above definition of the phase-space window, it is rather straightforward to integrate the $\mathcal{O}(\alpha^2)_{\rm prag}$ matrix element, keeping all terms within the $\mathcal{O}(\alpha^2)_{\rm prag}$ approximation. This we found insufficient for establishing a technical precision at the 0.03% level because some terms beyond $\mathcal{O}(\alpha^2)_{\rm prag}$ — especially for partially incomplete results — are of that size. We have therefore decided to integrate analytically up to the $\mathcal{O}(\alpha^3)_{\rm prag}$ terms[7]. In the $\mathcal{O}(\alpha^3)_{\rm prag}$ approximation we include by definition all terms from $\mathcal{O}(\alpha^2)_{\rm prag}$ plus terms of $\mathcal{O}(\alpha^3 L^3)$ and $\mathcal{O}(\alpha^2 L)$. In other words, terms of $\mathcal{O}(\alpha^2 L)$, due to our LL ansatz, and terms of $\mathcal{O}(\alpha^3 L^3)$, due to exponentiation, are integrated analytically over the entire phase space (within our academic trigger) exactly! The analytical integration involves many novel interesting calculational techniques and will be described in detail elsewhere [9]. The resulting integrated cross section is relatively simple and reads as follows:

$$
\begin{aligned}
&\sigma^{(2)}_B(t_{\min},t_{\max},V_{\max}) = \int_{t_{\min}}^{t_{\max}} dt \int_0^{V_{\max}} dV \;\; \rho^{(2)}_{\rm tot}(t,V),\\
&\rho^{(2)}_{\rm tot}(t,V) = b_0\; F(2\gamma)\; e^{2\Delta_{\rm YFS}(\gamma)}\; 2\gamma V^{2\gamma-1}\Big\{1+\gamma+\gamma^2/2\Big\}\\
&\quad +\; b_0\; F(2\gamma)\; e^{2\Delta_{YFS}(\gamma)}\; V^{2\gamma}\Big\{\gamma(-2+V) + \frac{\alpha}{\pi}\ln(1-V)\Big[-4+4V-2V^{-1}\Big]\\
&\quad -2\gamma^2 + \gamma^2\ln(1-V)\Big[3-(3/2)V-2V^{-1}\Big]\\
&\quad +\gamma^3(-7/4V) + \gamma^3\ln(1-V)\Big[5/4+(1/2)V-2V^{-1}\Big]\\
&\quad +\gamma^3\ln^2(1-V)\Big[-5/8+(5/16)V+(1/4)V^{-1}\Big] + \gamma^3{\rm Li_2(V)(2-V)}\\
&\quad +\gamma\frac{\alpha}{\pi}\Big[1/4+11V-(13/4)(2-V)^{-1}+(1/2)(2-V)^{-2}-6(2-V)^{-3}+2\sqrt{1-V}\Big]\\
&\quad +\gamma\frac{\alpha}{\pi}\ln(1-V)\Big[39/4-(19/4)V-2V^{-1}\\
&\qquad\qquad -2(2-V)^{-1}+(2-V)^{-2}-(1/2)(2-V)^{-3}-(3/2)\sqrt{1-V}\Big]\\
&\quad +\gamma\frac{\alpha}{\pi}\ln(1-V/2)\Big[-9/2+(3/4)V-4(2-V)^{-1}+2(2-V)^{-2}-4(2-V)^{-3}\Big] \qquad (15)\\
&\quad +\gamma\frac{\alpha}{\pi}\ln^2(1-V)\Big[19/8-(41/16)V+V^{-1}\Big] + \gamma\frac{\alpha}{\pi}\ln(1-V)\ln(V)(12-10V)\\
&\quad +\gamma\frac{\alpha}{\pi}\ln(1-V)\ln(2-V)(-1/2+(1/4)V) + \gamma\frac{\alpha}{\pi}\ln(1-V)\ln(V/2)(-6+5V)\\
&\quad +\gamma\frac{\alpha}{\pi}\ln\Big(1-V\Big)\ln(1-\sqrt{1-V}\Big)(-6+5V)\\
&\quad +\gamma\frac{\alpha}{\pi}\ln(2-V)\ln(1-V/2)(3/2-(11/4)V)\\
&\quad +\gamma\frac{\alpha}{\pi}\ln^2(1-V/2)(3/4-(5/8)V) + \gamma\frac{\alpha}{\pi}{\rm Li_2(1/2)(-3/2+(11/4)V)}\\
&\quad +\gamma\frac{\alpha}{\pi}{\rm Li_2\Big((1-V)/(2-V)\Big)(1/2-(1/4)V)} + \gamma\frac{\alpha}{\pi}{\rm Li_2\Big(1/(2-V)\Big)(1-(5/2)V)}\\
&\quad +\gamma\frac{\alpha}{\pi}{\rm Li_2\Big(-V/(2(1-V))\Big)(6-5V)} + \gamma\frac{\alpha}{\pi}{\rm Li_2\Big(1-1/\sqrt{1-V}\Big)(6-5V)}\\
&\quad -\gamma\xi\chi(V)/(1-V)\Big\},
\end{aligned}
$$

[7] This is probably the only known example of analytical integration over the full phase space up to three photons.

where $\gamma = 2(\alpha/\pi)(L-1)$, $b_0 = \chi(\xi)$, $\chi(x) \equiv [1+(1-x)^2]/2$, $\xi = |t|/s$, $F(x) \equiv \exp(-Cx)/\Gamma(1+x)$, and $\Delta_{\rm YFS}(\gamma) = \gamma/4 - (\alpha/\pi)(1/2+\pi^2/6)$. (Here, C is Euler's constant.) Note that the $\mathcal{O}(\alpha^2)_{\rm prag}$ part of the formula is very compact (first three lines) and that its LL content is functionally identical to the non-singlet, second-order, structure function of the photon in the electron[8]. The terms of $\mathcal{O}(\alpha^3 L^3)$ and $\mathcal{O}(\alpha^2 L)$, which represent most of the formula, finally turn out to be numerically small (in fact below 0.04%), provided the YFS factor $F(2\gamma)\, e^{2\Delta_{\rm YFS}}\, V^{2\gamma-1}$ is not expanded. In Fig. 1a we show a comparison of Eq. (15) with the Monte Carlo BHLUMI 4.01. Although the analytical integration over $|t|$ and V is feasible, we do it numerically with the help of the standard Gauss technique[9]. As we see, the Monte Carlo and the semi-analytical results differ by up to 0.03%. We conclude that for our "academic trigger" we have obtained a 0.03% technical precision.

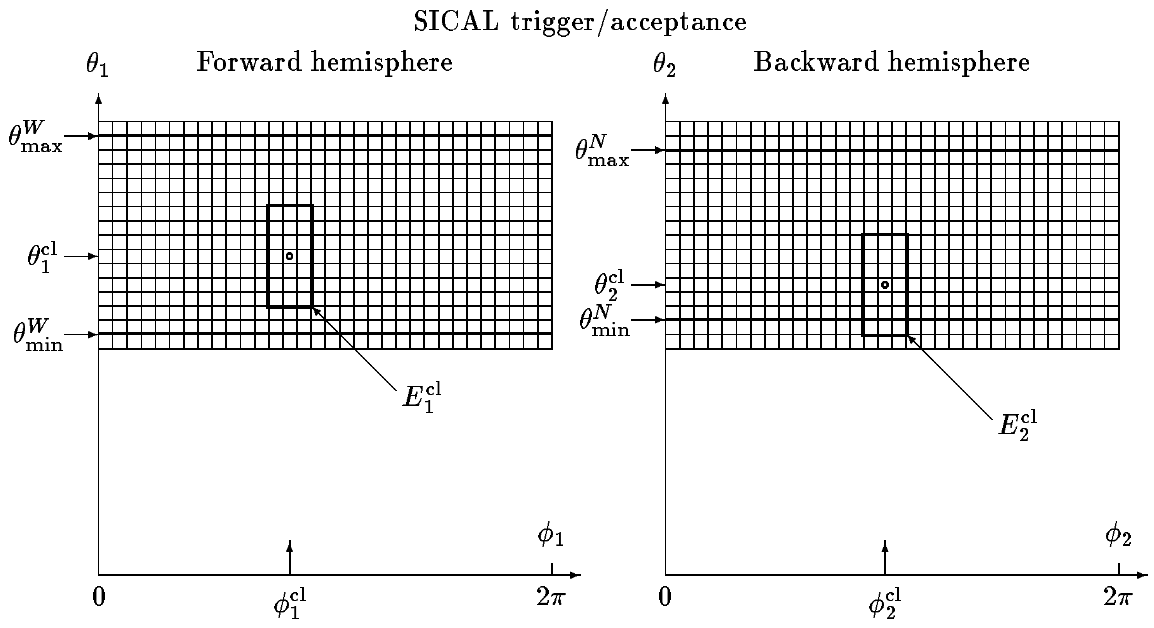
Our main numerical results and discussion of the total theoretical error will be presented for a realistic example of the experimental set of cuts, called shortly a *trigger*. We shall use a trigger representing the silicon luminometer SICAL of the ALEPH Collaboration, which is quite typical of all LEP experiments. The main feature of this trigger is a calorimetric detection of the final-state photons and electrons, i.e. the detector cannot tell electrons from photons — only the total energy is registered. In Fig. 2 we show in detail how electrons and photons are combined in the scattering solid angle into a "cluster". Each of the clusters in the opposite hemispheres is required to take away a substantial part of the total energy, typically more than half of it. This "energy cut" is parametrized using a combination of $U_{\rm min}, Z_{\rm min}$ and $Y_{\rm min}$ parameters defined in Fig. 2. The centre of each cluster is required to be within a certain θ range. In Fig. 2 we show the so-called "asymmetric θ acceptance" which is used by all LEP experiments for technical reasons[10].

In the following we shall show numerical results from the MC event generator BHLUMI 4.01 for the SICAL trigger and discuss the total theoretical error of these results. Let us first show the difference of the integrated cross section between BHLUMI 2.01 [3] with YFS exponentiation using the $\mathcal{O}(\alpha^1)$ matrix element and the new BHLUMI 4.01 with YFS exponentiation based on the $\mathcal{O}(\alpha^2)_{\rm prag}$ matrix element as defined in eqs. (1–12). The difference divided by Born cross section is shown in Fig. 3 as a function of one of the energy cut parameters. This difference should represent mainly the missing $\mathcal{O}(\alpha^2 L^2)$ in BHLUMI 2.01. The actual value of -0.025% for the standard SICAL acceptance (marked with the vertical line) is surprisingly small, if we remember that the relevant order parameter $2(\alpha/\pi)^2 L^2 \simeq 0.25\%$, see also Ref. [5], and that in Ref. [2] we have estimated this difference to be less than 0.15%. This result simply means that the YFS exponentiation in BHLUMI 2.01 sums up LL higher orders in an extremely efficient way! See also Ref. [10] for a similar phenomenon. On the other hand the difference is large enough on the scale of the experimental LEP luminometry precision (below 0.1%) to

[8]This is due to the fact that the variable V in the LL has a very simple meaning.

[9]The integrand is a very smooth function, suitable for this method, if peaks are removed either by change of variables or subtraction.

[10]The asymmetric cut helps to reduce the uncertainty due to the unknown position of the interaction point.



$$\theta^W_{\rm min} < \theta^{\rm cl}_1 < \theta^W_{\rm max}$$
$$\theta^N_{\rm min} < \theta^{\rm cl}_2 < \theta^N_{\rm max}$$
$$|\phi^{\rm cl}_1 - \phi^{\rm cl}_2| < \Delta\phi_{\rm max}$$

$$\min(E^{\rm cl}_1, E^{\rm cl}_2) > U_{\rm min} E^2_{\rm beam}$$
$$E^{\rm cl}_1 E^{\rm cl}_2 > Z_{\rm min} E_{\rm beam}$$
$$(E^{\rm cl}_1 + E^{\rm cl}_2)/2 > Y_{\rm min} E_{\rm beam}$$

Figure 2: Geometry and acceptance of SICAL luminosity subdetector in the ALEPH experiment. Each side detector consists of 16×32 pads. A single pad measures the total energy of electrons and photons. A pad with the maximum energy together with its 3×7 neighbourhood is called a cluster. The total energy registered in the cluster is denoted by $E^{\rm cl}_i$ and the average position (weighted with energy) is denoted by $(\theta^{\rm cl}_i, \phi^{\rm cl}_i)$, $i = 1, 2$. By definition $\phi_1 = \phi_2$ for back-to-back particles. We depict the asymmetric type of angular acceptance; thick lines limit the wide/narrow θ-range for forward/backward hemisphere Pads of the cluster which spill over the angular range (thick lines) are used to determine total energy and average position of the cluster (see the example of the backward hemisphere).

justify replacing BHLUMI 2.01 by BHLUMI 4.01 to calculate the cross section.

The dominant contribution to the total theoretical error of BHLUMI 2.01 in Ref. [2] was the missing $\mathcal{O}(\alpha^2 L^2)$ plus technical precision. It was found to be 0.15%. The source of this number was the difference between the exponentiated $\mathcal{O}(\alpha^1)$ cross section and the "hybrid" Monte Carlo event generator consisting of the $\mathcal{O}(\alpha^1)$ OLDBIS and the $\mathcal{O}(\alpha^3 L^3)$ LUMLOG (both included in the BHLUMI 2.01 set). In Fig. 4, we show again the same difference, but now using BHLUMI 4.01, with the YFS-exponentiated $\mathcal{O}(\alpha^2)_{\rm prag}$ matrix element defined in this paper, instead of the previous YFS-exponentiated $\mathcal{O}(\alpha^1)$ matrix element. With the very high statistics of 2×10^9 events (a factor 100 higher than in Ref. [2]) we find that the difference (divided by Born) for the SICAL trigger is again well within 0.15%. Since this result is a cornerstone of our total theoretical error estimate, let us discuss it in more detail. In Ref. [2] this difference was interpreted as the

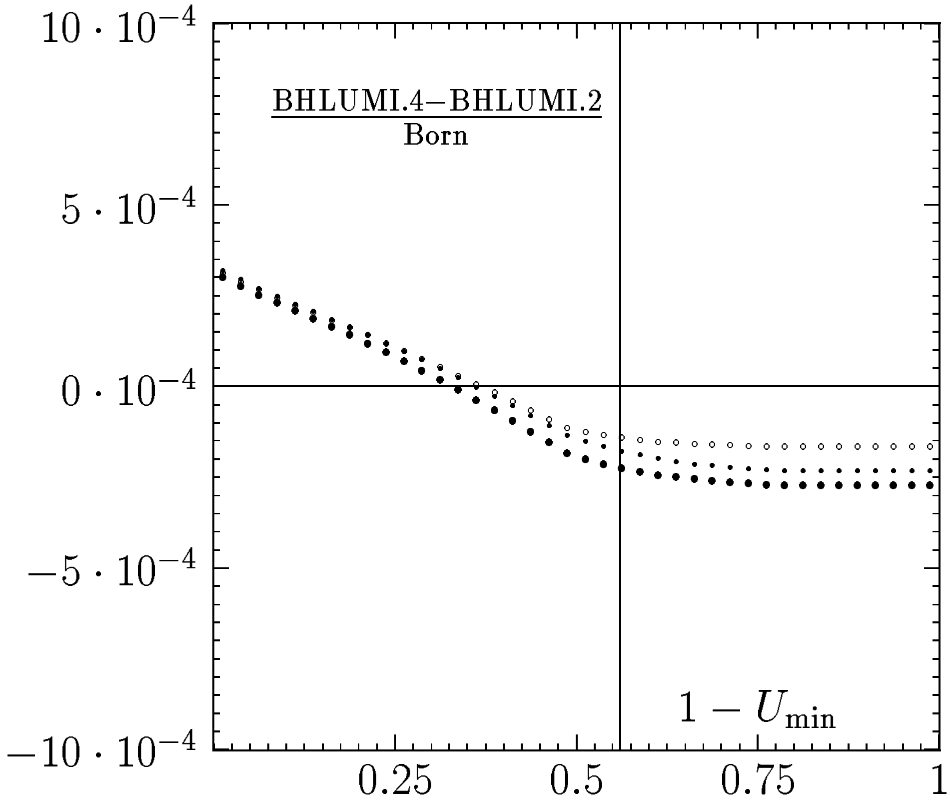


Figure 3: For the ALEPH SICAL detector we plot c_{4-2} = (BHLUMI.4 − BHLUMI.2)/Born as a function of the energy cut $1 - U_{\rm min}$, where $U_{\rm min} = \min(E_1^{\rm cl}, E_2^{\rm cl})/E_{\rm beam}$. The standard value $(1-U_{min})^{CUT} = 0.561341$ is marked with the vertical line. Three curves plotted with small dots, open circles and big dots represent angular cuts W-W, N-N and N-W, respectively, where W and N denote wide or narrow angular ranges on one side of the detector. The wide (W) angular range is $\theta_A + \Delta < \theta_1^{\rm cl} < \theta_B - \Delta$, and the narrow (N) angular range is $\theta_A + 2\Delta < \theta_2^{\rm cl} < \theta_B - 4\Delta$, where $\theta_A = 0.024$, $\theta_B = 0.058$ and $\Delta = (\theta_B - \theta_A)/16$. The other cuts are: (a) auxiliary energy cuts $Y_{\rm min} = 0.60315$, $Z_{\rm min} = 0$, (b) acoplanarity cut $\Delta\phi_{\rm max} = 0.52359$, see Fig. 2 for cut-off definitions.

missing $\mathcal{O}(\alpha^2 L^2)$ term plus technical precision. What is it now? At present it should be interpreted as missing $\mathcal{O}(\alpha^2 L)$ terms and $\mathcal{O}(\alpha^3 L^3)$ plus technical precision (for a given realistic trigger). As we already know from Ref. [2], see also discussion and numerical results in Ref. [5], the missing $\mathcal{O}(\alpha^3 L^3)$ in BHLUMI 4.01 is negligible, below 0.02%, thanks to the YFS exponentiation which sums up higher-orders LL very efficiently. We conclude, therefore, that the difference in Fig. 4 is mainly the missing $\mathcal{O}(\alpha^2 L)$ terms in OLBIS+LUMLOG compared to those generated by YFS exponentiation in BHLUMI 4.01 plus technical precision. Does 0.15% look reasonable as an estimate for these missing $\mathcal{O}(\alpha^2 L)$ terms in the presence of the realistic experimental cuts? At first sight it looks as an

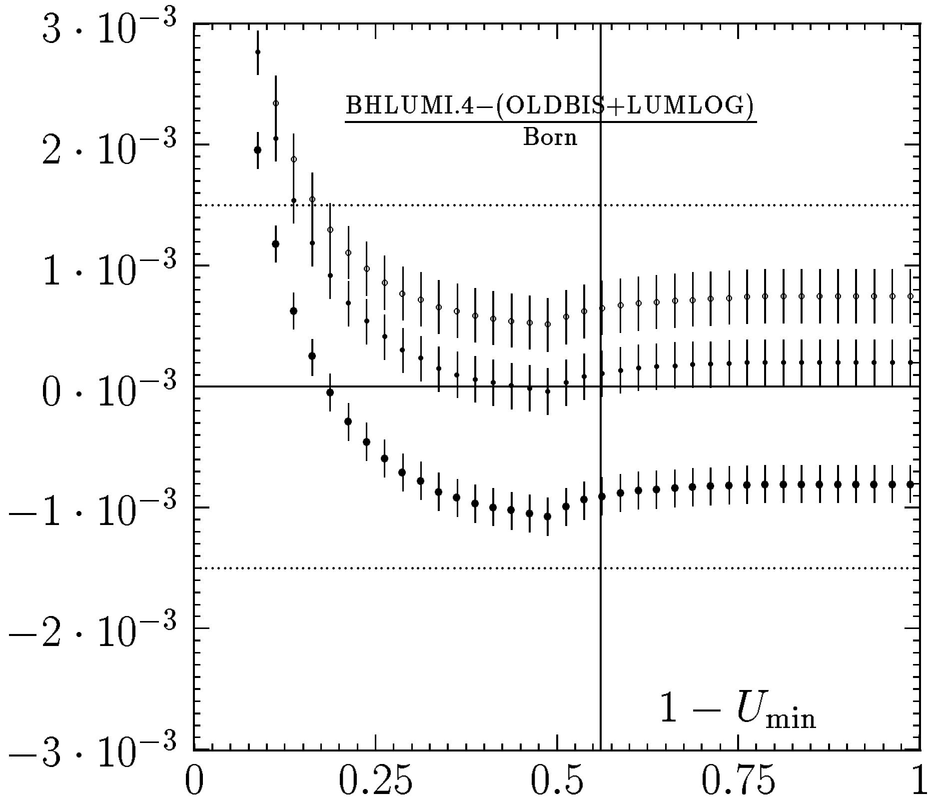


Figure 4: The difference d_3 = (BHLUMI.4 – OLDBIS – LUMLOG)/Born for the ALEPH SICAL detector as a function of the energy cut $1 - U_{\rm min}$. Cuts are the same as in Fig. 3.

overestimate because the order parameter $2(\alpha/\pi)^2 L = 0.02\%$. On the other hand such a contribution may be enhanced by the logarithm of the cut-off parameters. In the situation when we do not calculate the $\mathcal{O}(\alpha^2 L)$ contribution but we only estimate it, the best is to rely on the method which is cut–off–dependent — the difference in Fig. 4 is precisely of this type, in the plot we vary the angular acceptance and the energy cut. In fact we have varied other cut parameters as well, i.e. the size of the cluster or the acoplanarity cut. We have also introduced an acollinearity cut. The difference plotted in Fig. 4 remained all the time within 0.15%. This leaves open the question of the size of the missing $\mathcal{O}(\alpha^2 L)$ terms from BHLUMI 4.01 that are not generated by YFS exponentiation — comparison with OLDBIS+LUMLOG, which only has exact $\mathcal{O}(\alpha)$+$\mathcal{O}(\alpha^2 L^2)$+$\mathcal{O}(\alpha^3 L^3)$, can say nothing about these terms. They can be estimated, in the absence of an actual calculation, following an optimistic version of the methods which we used in Ref. [2] as $\sim 0.6\times$(missing $\mathcal{O}(\alpha^2 L^2)$ in BHLUMI 2.01 not generated by YFS exponentiation)$= -0.015\%$ (as in Table 3 in Ref. [2]). In view of the possible technical precision issue in the difference, between BHLUMI 4.01 and OLDBIS+LUMLOG, we may conservatively use 0.15% as an estimate of the total precision on the bremsstrahlung corrections in BHLUMI 4.01 up to this point in our discussion.

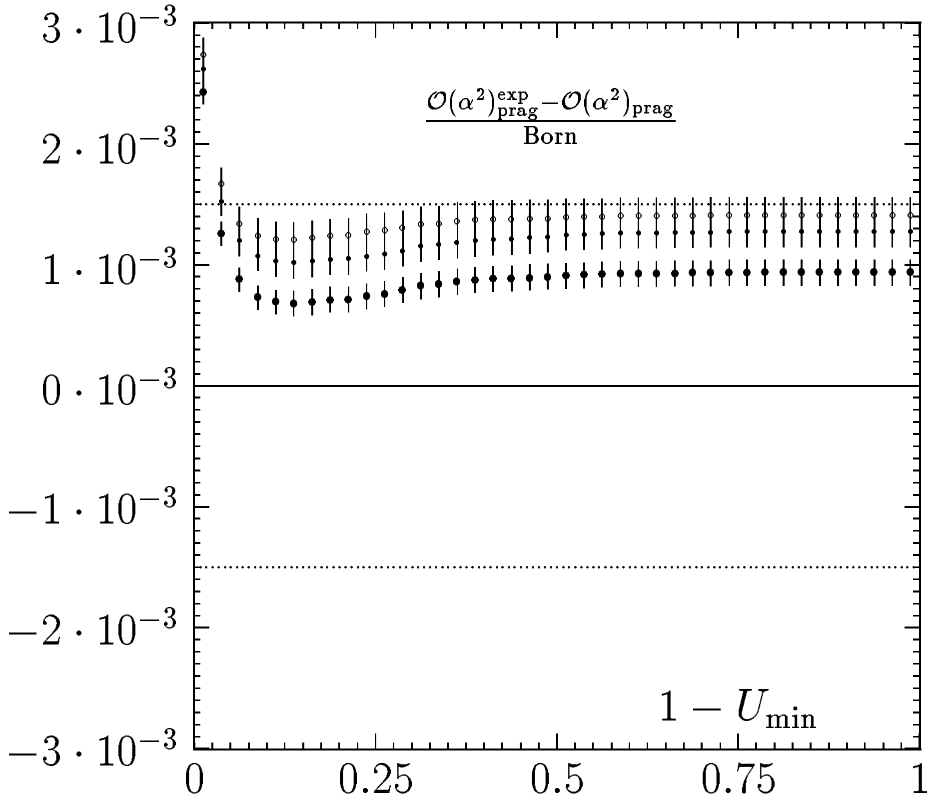


Figure 5: Difference of the cross section calculated with the $\mathcal{O}(\alpha^2)_{\rm prag}$ matrix element with YFS exponentiation and without exponentiation for the SICAL trigger. The difference divided by the Born cross section is plotted as a function of the energy cut $1 - U_{\rm min}$. The other cuts are the same as in Fig. 3.

The other possibility to get an idea of the magnitude of the $\mathcal{O}(\alpha^2 L)$ missing contribution is to stay with the same (realistic) trigger and to vary the matrix element within $\mathcal{O}(\alpha^2)_{\rm prag}$ and/or to switch on/off YFS exponentiation. We did a lot of exercises of this type, see Ref. [5] for some of the results, and the largest effect of this kind we could see is presented in Fig. 5, where we plot the difference between the cross section of $\mathcal{O}(\alpha^2)_{\rm prag}$ with and without the YFS exponentiation for the SICAL trigger. The difference is also within 0.15%. In the case of the "academic" trigger, we have analysed the above difference in detail using explicit analytical expressions. We have found that for a loose energy cut the difference is saturated by $\mathcal{O}(\alpha^3 L^3)$ because it is missing in the calculation without YFS exponentiation, while for a strong cut it is dominated by $\mathcal{O}(\alpha^2 L)$ artefacts (logs of cut-off parameters) in the calculation without exponentiation. The situation is most probably the same in the realistic trigger, but for the moment we cannot prove it. Nevertheless the above result is interesting, especially in that there are some general similarities between the plots in Figs. 4 and 5. We summarize that the conservative upper limit of 0.15% for the missing $\mathcal{O}(\alpha^2 L)$ plus technical precision is very well established for the SICAL trigger[11]. The actual number may change for another kind of trigger and the exercises with cut-off variation have to be repeated each time.

BHLUMI 4.01 precision		
Type of contribution	Physical	Technical
{1} Vacuum polarization hadronic	0.05%	
{2} Missing part of $\left(\frac{\alpha}{\pi}\right)^2 L$ brems.	0.15%	
{3} Missing part of $\left(\frac{\alpha}{\pi}\right)^3 L^3$ brems.	0.000%	0.008%
{4} Up–down interference	0.004%	0.001%
{5} e^+e^- pairs LL non-singlet	0.007%	0.002%
{6} e^+e^- pairs LL singlet	$< 0.002\%$	
{7} pairs NLL, quarks, μ	$< 0.007\%$	
{8} Z exchange	0.03%	0.03%
{9} γ exchange s-channel	$< 0.0002\%$	
Total	0.16%	

Table 1: The precision on the cross section at $\sqrt{s} = 91.187$ GeV for the SICAL-type acceptance. The precision listed is for BHLUMI 4.01 [8], with the understanding that the Z-exchange correction is taken from Refs. [11, 12] and pairs effects are taken from Refs. [2, 13]. The contributions {1} and {9} are included in BHLUMI 4.01 in the same way as in BHLUMI 2.01 [3]. The error of {1} for the present SICAL angular range is taken from Refs. [14, 11, 12]. The correction {4} is not included in the multiphoton BHLUMI 4.01 but is calculated in first order using the OLDBIS sub-generator (included in BHLUMI 4.01) much as in Ref. [2]. The contributions {5–8} are not included directly in the multiphoton BHLUMI 4.01, but they are known from other sources together with their uncertainties: {5,6,7} are recalculated following Refs. [2, 13] and the error of {8} is taken from Refs. [14, 11, 12]. The total theoretical error is indicated.

In order to obtain the total theoretical error for the example of the SICAL detector we combine the above error with errors due to vacuum polarization, QED corrections to the $\gamma_t \otimes Z_s$ contribution and other less important errors — they are recalculated or taken from the published works. The list of error contributions with the complete set of references is shown in Table 1. The total theoretical error of the low-angle Bhabha cross section for the SICAL-type experimental cut we estimate, conservatively, to be 0.16%.

In recent years other groups have also made considerable efforts in the area of the calculation of the QED corrections to the LABH cross section. Due to limited space, let us refer the reader to the relevant papers [15, 16, 17, 18, 19, 20]. The important differences between our work and other works are that (a) we implement our calculation in the form of the full MC event generator, i.e. we are able to calculate the cross section

[11]We have done the same exercises for the OPAL SiW detector with the same 0.15% result.

for *arbitrary detector acceptance* and vary easily each cut-off parameter; (b) thanks to this we are able to estimate the physical precision for any given experimental cut (taking into account cut-off-dependent enhancement factors); (c) our precision estimate includes also the technical precision, a very important component in the total error, often neglected in other works; and (d) the exponentiation procedure, which very efficiently sums up higher-order contributions, is done at the level of the differential cross section (YFS scheme) and not at the level of the integrated[12] inclusive distributions (naïve exponentiation).

Our main results can be summarized as follows:

- The $\mathcal{O}(\alpha^2)_{\rm prag}$ exponentiated matrix element in the new version of BHLUMI 4.01 is implemented and we have a lot of evidence that it was done correctly.
- A technical precision as high as 3×10^{-4} has been established for BHLUMI 4.01 for the special "academic" type of trigger (cut-offs).
- We estimate conservatively, the dominant $\mathcal{O}(\alpha^2 L)$ photonic QED contribution to the total precision to be 0.15%, which provides us with the new BHLUMI 4.01 total bremsstrahlung uncertainty of 0.16% for the example of the SICAL detector of ALEPH. This is a considerable improvement over the previously published value of 0.24%.

The future steps toward a conservative precision below 0.1% are: getting a better estimate of the technical precision for every set of experimental triggers (not only for the "academic" trigger), inclusion of the photonic $\mathcal{O}(\alpha^2 L)$ corrections in the full exponentiated matrix element, reintroduction of the $\gamma_t \otimes Z_s$ interference and inclusion (optionally) of the incomplete $\mathcal{O}(\alpha^3 L^3)$ corrections in LUMLOG (corresponding to the BHLUMI matrix element) such that for arbitrary cuts we have an ultimate control on the (small) $\mathcal{O}(\alpha^3 L^3)$ correction missing in the BHLUMI cross section. Finally, let us strongly stress the importance of the comparisons of the fully independent calculations from different groups of authors. For the low-angle Bhabha process the first successful comparisons of this kind were shown in the framework of the recent LEP workshop on precision calculations for the Z resonance, see ref. [21][13]. This work is continued in the present Bhabha subgroup of the LEP II workshop. Preliminary encouraging results suggest that the level of agreement among various computer codes below 0.1% can be reached.

Acknowledgements

Useful discussions with Drs. S. Arcelli, M. Dallavalle, B. Pietrzyk, and W.D. Schlatter are acknowledged. The authors wish to acknowledge the support and kind hospitality of the CERN Theory Division; two of the authors (SJ and BFLW) wish to express gratitude for the support of the ALEPH Collaboration, one (SJ) for that of the IN2P3 Franco-Polish Collaboration through LAPP Annecy, and one (SJ) for the Italian INFN through

[12]In the naïve exponentiation the integration over the phase space before the exponentiation (i.e. resummation of the infrared correction to infinite order) inhibits free manipulation of the cut-offs.

[13]See also `http://preprints.cern.ch/yellow/1995/95-03/contents.html` under World Wide Web.

Bologna University. This work was supported partly by Polish Government grant KBN 2P30225206, and by the US DoE contract DE-FG05-91ER40627.

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