UTHEP–92–0701
July 1992

Analytical Results for Low Angle Bhabha Scattering with Pair Production [1]

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ABSTRACT

We calculate the effect of pair production on the SLC/LEP luminosity process of low angle Bhabha scattering. We work in the leading logarithm approximation to orders α^2 and α^3 for the nonsinglet and the singlet contributions, respectively. In particular we calculate the size of singlet contribution to be below 0.007% for the energy cuts $x_c > .1$, as it was estimated in Ref. [1]. In this way, we remove the contribution of such pairs from the theoretical uncertainty in the SLC/LEP luminosity calculation of Jadach et al. via the Monte Carlo BHLUMI2.01 at the precision level of .01%. Results are illustrated for typical SLC/LEP acceptance cuts.

[1]Supported in part by the US Department of Energy Contracts DE -FG05-91ER40627 and DE-AC03-76ER00515, and by the Polish Ministry of Education grants KBN 223729102 and KBN 203809101.

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1 Introduction

The era of the high precision Z^0 physics has now progressed to the below 1% regime [2]. The current projection is that the accumulated Z^0 number will exceed 10^6 per LEP experiment in the near future. Accordingly, new experimental apparati are under preparation which will exploit the respective .1% statistics [3]. Such apparatus will allow, for example, the basic SLC/LEP luminosity process, low angle Bhabha scattering, to be measured at LEP with an experimental systematic error of .15%. Accordingly, it is important to improve the theoretical precision of the calculation of this process from its current .25% [1] to the .05% regime. In this paper , we treat one of the significant contributors to the .25% error estimate in Ref. [1], namely, the effect of the production of pairs on the basic low angle Bhabha scattering process.

Specifically, the effect of the pair production on the s-channel processes $e^+e^- \to f\bar{f}$, $f \neq e$, has been analyzed in Ref. [4]. What we do in our work is to extend the results of two of us (S.J. and M.S.) in Ref. [4] to the t-channel dominated low angle Bhabha scattering process. In this way, we arrive at *new* $\mathcal{O}(\alpha^2)$ and $\mathcal{O}(\alpha^3)$ leading logarithmic (LL) formulae for the effect of such pair production on the latter process. Such formulae have not appeared elsewhere.

In order to remain consistent with the precision standard which we set in Refs. [1, 6], we will specify the technical and physical precisons of our work as they are defined in Ref. [6][3]. The net result is that the contribution of pair production to the uncertainty on the SLC/LEP luminosity cross sections in Refs. [1, 6] is now removed and this pair production effect may be removed from the data in complete analogy with our removal of the $O(\alpha^2)$ bremsstrahlung effect in Ref. [6]. Alternatively, the pair production may be incorporated directly into the Monte Carlo program BHLUMI 2.01 [1], which currently simulates the SLC/LEP luminosity process to .25% precision. A prototype version of this latter incorporation of pairs into BHLUMI 2.01 exists [7] and the complete implementation of such pair production will appear elsewhere [7]. Thus, with our work in this paper, an important step toward the .05% precision regime for the theoretical uncertainty on the basic LEP/SLC luminosity process has been completed. In addition, our formulae are of general theoretical interest in their own right and they corroborate

[3] As defined in [6] *technical precision* includes machine rounding errors, programming bugs, approximations in matrix elements and phase space integration, residual dependence on infrared regulators, etc., while physical precision includes higher orders, missing diagrams, etc.

the error estimates made in Refs. [1, 6] for the effects of pairs on the SLC/LEP luminosity cross section.

Our work in this paper is organized as follows. In the next Section, we present our formulae for low angle Bhabha scattering with pair production. In the Sect. 3, we illustrate these formulae with results for the typical LEP/SLC type luminosity cuts and give a recipe for including the effects of pairs in the LEP/SLC luminosity analysis. Sect.4 contains some concluding remarks.

2 Leading-Logarithmic analytical formulae for low angle Bhabha scattering with pair production

The master LL, structure function-type, formula for low angle Bhabha scattering with the initial state corrections reads [6, 5]

$$\begin{aligned} \sigma_{LL}^{asym} &= \sigma_{LL}(\xi_{mi}^{(1)}, \xi_{ma}^{(1)}, \xi_{mi}^{(2)}, \xi_{ma}^{(2)}, x_c) \\ &= \int_0^1 dx_1 dx_2 D(x_1,\beta) D(x_2,\beta) \int d\xi \frac{d\sigma^B}{d\xi} \Theta_{\xi_{mi}^{(1)}}^{\xi_{ma}^{(1)}}(\xi_1) \Theta_{\xi_{mi}^{(2)}}^{\xi_{ma}^{(2)}}(\xi_2) \Theta(x_1 x_2 - x_c). \end{aligned} \tag{1}$$

The $D(x, \beta)$ are the electron structure functions, either nonsinglet (NS) or singlet (S). The recent review of formulae for D^{NS} and D^S can be found in [8, 9]. The effective LL parameter β can be defined in a few ways differing by subleading terms, see eg. Ref. [6] for details. In the following we use the simplest form of β, independent of ξ

$$\beta = \frac{2\alpha}{\pi} \ln \frac{s\xi_{mi}}{m_{el}^2}. \tag{2}$$

The singlet function D^S itself describes a possible e^+e^- pair production mechanism.To account the production via the nonsinglet mechanism the parameter β of eq. (2) should be replaced by the running one

$$\beta_R = -6 \ln\Big(1 - \frac{\alpha}{3\pi} \ln \frac{s\xi_{mi}}{m_{el}^2}\Big). \tag{3}$$

The energy cut is denoted by x_c and the trigger cuts are defined by $\Theta_{\xi_{mi}^{(i)}}^{\xi_{ma}^{(i)}}(\xi_i) = \Theta(\xi_{ma}^{(i)} - \xi_i)\Theta(\xi_i - \xi_{mi}^{(i)})$, $i = 1, 2$ where

$$\xi_1 = x_2\xi / \Big(x_2\xi + x_1(1-\xi)\Big), \qquad \xi_2 = x_1\xi / \Big(x_1\xi + x_2(1-\xi)\Big). \tag{4}$$

The Born cross section reads
$$\frac{d\sigma^B}{d\xi} = \frac{2\pi\alpha^2}{sx_1x_2}\frac{1+(1-\xi)^2}{\xi^2}. \tag{5}$$

Evaluation of the master integral (1) is done in a few steps.

(i) The trigger cuts can be solved for, say, x_2
$$\Theta^{\xi^{(1)}_{ma}}_{\xi^{(1)}_{mi}}(\xi_1): \qquad x_1\frac{1-\xi}{\xi}\frac{\xi^{(1)}_{mi}}{1-\xi^{(1)}_{mi}} \le x_2 \le x_1\frac{1-\xi}{\xi}\frac{\xi^{(1)}_{ma}}{1-\xi^{(1)}_{ma}}, \tag{6}$$
$$\Theta^{\xi^{(2)}_{ma}}_{\xi^{(2)}_{mi}}(\xi_2): \qquad x_1\frac{\xi}{1-\xi}\frac{1-\xi^{(2)}_{ma}}{\xi^{(2)}_{ma}} \le x_2 \le x_1\frac{\xi}{1-\xi}\frac{1-\xi^{(2)}_{mi}}{\xi^{(2)}_{mi}}. \tag{7}$$
At this point we make a simplification and put both limits for ξ_1 and ξ_2 identical, i.e. $\xi^{(1)}_{ma,mi} = \xi^{(2)}_{ma,mi} = \xi_{ma,mi}$. At the very end of the calculation we will get rid of this restriction. Now, conditions (6) and (7) can be evaluated to a compact form
$$ax_1 \le x_2 \le \frac{1}{a}x_1, \qquad a = a(\xi) = \max\Bigl(\frac{\xi}{1-\xi}\frac{1-\xi_{ma}}{\xi_{ma}}, \frac{1-\xi}{\xi}\frac{\xi_{mi}}{1-\xi_{mi}}\Bigr). \tag{8}$$
Let us note that always $a \le 1$ and also $\xi_{mi} \le \xi \le \xi_{ma}$.

(ii) We introduce a new integration variable x: $1 = \int\limits_0^1 dx\delta(x - x_1x_2)$ and rewrite the master formula (1) as
$$\begin{aligned}\sigma^{sym}_{LL} &= \sigma_{LL}(\xi_{mi}, \xi_{ma}, x_c) \\ &= \int\limits_{x_c}^{1} dx \int\limits_{\xi_{mi}}^{\xi_{ma}} d\xi \frac{d\sigma^B}{d\xi} \int\limits_0^1 dx_1 \int\limits_{ax_1}^{\min(1,x_1/a)} dx_2 D(x_1,\beta)D(x_2,\beta)\delta(x - x_1x_2).\end{aligned} \tag{9}$$
The integration domain of variables x_1-x_2 is shown in Fig. 1. Integration goes along the hyperbolas $x_1x_2 = x$. This domain in a natural way separates into two parts, depending on the value of x. For $x \ge a$ we get $x \le x_1 \le 1$ whereas for $x < a$ we have $\sqrt{ax} \le x_1 \le \sqrt{x/a}$. This way, integrating out the δ function,we arrive at
$$\sigma^{sym}_{LL} = \int\limits_{x_c}^{1} dx \int\limits_{\xi_{mi}}^{\xi_{ma}} d\xi \frac{d\sigma^B}{d\xi}\Bigl[\Theta(x \ge a)I_2(x,\beta) + \Theta(a > x)I_1(x,a,\beta)\Bigr], \tag{10}$$
$$I_1(x,a,\beta) = \int\limits_{\sqrt{ax}}^{\sqrt{x/a}} \frac{dx_1}{x_1}D(x_1,\beta)D(\frac{x}{x_1},\beta), \tag{11}$$
$$I_2(x,\beta) = \int\limits_{x}^{1} \frac{dx_1}{x_1}D(x_1,\beta)D(\frac{x}{x_1},\beta). \tag{12}$$

Let us notice some important properties of integrals (11) and (12). The I_2 function depends only on x and is independent of ξ (or a equivalently). It means that the angular integral over $d\xi$ of σ^B factorizes and can be easily done. Moreover, the I_2 integral is simply the well known convolution of two structure functions and third order formulae for both nonsinglet and singlet functions can be found in [8, 9]. The I_1 function is not the same case at all. It depends on both x and ξ. However, another simplification appears here. Both the infrared points $x_1 = 1$ and $x_2 = 1$ are excluded from the integration domain of I_1. It means that I_1 is non-zero only when at least two real photons/pairs are emitted, one by each incoming particle. That makes the I_1 to be at least $\mathcal{O}(\alpha^2)$ for the nonsinglet function and $\mathcal{O}(\alpha^3)$ for the singlet function.

(iii) In the next step we deal with the remaining dx-$d\xi$ integrations. We introduce the new variable $z = (1-\xi)/\xi$, and eq. (10) becames

$$\sigma_{LL}^{sym} = \int\limits_{x_c}^{1} dx \int\limits_{z_{mi}}^{z_{ma}} dz \frac{1}{x}\frac{d\bar{\sigma}(z)}{dz}\Big[\Theta(x \geq a) I_2(x,\beta) + \Theta(a > x) I_1(x,a,\beta)\Big], \tag{13}$$

where $d\bar{\sigma}(z)/dz = (2\pi\alpha^2/s)\Big[1 + \Big(1 - 1/(1+z)\Big)^2\Big]$, $z_{mi} = (1-\xi_{ma})/\xi_{ma}$, $z_{ma} = (1-\xi_{mi})/\xi_{mi}$ and $a = a(z) = \max(z_{mi}/z, z/z_{ma})$. Note, that $\sqrt{z_{mi}/z_{ma}} \leq a \leq 1$. The x-z phase-space is shown in Fig. 2. The value of z for which a changes is $\sqrt{z_{mi} z_{ma}}$ and consequently, the curves dividing phase-space into $x < a$ (integral I_1) and $x \geq a$ (integral I_2) regions are $x = z/z_{ma}$ in the upper part and $xz = z_{mi}$ in the lower part. Making use of the specific form of the Born cross section $\sigma^B(x,z) = \bar{\sigma}(z)/x$ we can rewrite our master formula (13) as follows

$$\begin{aligned}
\sigma_{LL}^{sym} &= \int\limits_{\max(x_c,\sqrt{z_{mi}/z_{ma}})}^{1} dx I_2(x,\beta)\frac{1}{x}\bar{\sigma}(x) + \int\limits_{z_{mi}}^{\min(z_{mi}/x_c,\sqrt{z_{mi}z_{ma}})} dz \frac{d\bar{\sigma}(z)}{dz} J_1(a = \frac{z_{mi}}{z},\beta) \\
&+ \int\limits_{\max(z_{ma}x_c,\sqrt{z_{mi}z_{ma}})}^{z_{ma}} dz \frac{d\bar{\sigma}(z)}{dz} J_1(a = \frac{z}{z_{ma}},\beta)
\end{aligned} \tag{14}$$

where

$$\begin{aligned}
\bar{\sigma}(x) &= \int\limits_{z_{mi}/x}^{xz_{ma}} dz \frac{d\bar{\sigma}(z)}{dz} \\
&= \frac{2\pi\alpha^2}{s}\Big(2xz_{ma} - 2\frac{z_{mi}}{x} - 2\ln\frac{1+xz_{ma}}{1+\frac{z_{mi}}{x}} - \frac{1}{1+xz_{ma}} + \frac{1}{1+\frac{z_{mi}}{x}}\Big)
\end{aligned} \tag{15}$$

and

$$J_1(a,\beta) = \int_{x_c}^{a} dx \frac{1}{x} I_1(x,a,\beta). \tag{16}$$

Eq. (14) is the final form of the master LL equation (1). The remaining one dimensional integration in eq. (14) will be performed numerically. The explicit form of J_1 depends on the structure functions used and will be discussed later on for each function separately. An interesting remark results from Fig. 1. If the energy cut-off x_c falls below $\sqrt{z_{mi}/z_{ma}}$ the only additional contribution to σ_{LL} comes from the I_1 function (two hard photons), which in turn starts at second order in β. As a consequence the dependence of σ_{LL} on x_c should be remarkably weaker than for $x_c > \sqrt{z_{mi}/z_{ma}}$. Indeed it is clearly visible in any figure of Refs. [6, 1].

(iv) Finally, we can restore asymmetric cuts, i.e. put $\xi^{(1)}_{ma,mi} \neq \xi^{(2)}_{ma,mi}$. The following remark is crucial here. The basic formula (1) is invariant with respect to the exchange of e^+ and e^- triggers, i.e. exchange $\xi_1 \leftrightarrow \xi_2$ of eq. (4)

$$\sigma^{asym}_{LL}(\xi^{(1)}_{mi},\xi^{(1)}_{ma},\xi^{(2)}_{mi},\xi^{(2)}_{ma},x_c) = \sigma^{asym}_{LL}(\xi^{(2)}_{mi},\xi^{(2)}_{ma},\xi^{(1)}_{mi},\xi^{(1)}_{ma},x_c). \tag{17}$$

This is true since the replacement $\xi_1 \leftrightarrow \xi_2$ is equivalent to $x_1 \leftrightarrow x_2$, compare eq. (4), and the rest of formula (1) is symmetric with respect to x_1 and x_2.Assuming, conventionally, that $\xi^{(2)}_{mi} < \xi^{(1)}_{mi} < \xi^{(1)}_{ma} < \xi^{(2)}_{ma}$ we can therefore rewrite any asymmetric cross section in terms of symmetric ones, see also Fig. 3

$$\begin{aligned}\sigma^{asym}_{LL}(\xi^{(1)}_{mi},\xi^{(1)}_{ma},\xi^{(2)}_{mi},\xi^{(2)}_{ma},x_c) &= \frac{1}{2}\Big[\sigma^{sym}_{LL}(\xi^{(1)}_{mi},\xi^{(2)}_{ma},x_c) - \sigma^{sym}_{LL}(\xi^{(1)}_{ma},\xi^{(2)}_{ma},x_c) \\ &+ \sigma^{sym}_{LL}(\xi^{(2)}_{mi},\xi^{(1)}_{ma},x_c) - \sigma^{sym}_{LL}(\xi^{(2)}_{mi},\xi^{(1)}_{mi},x_c)\Big].\end{aligned} \tag{18}$$

This completes the evaluation of general LL formula for Bhabha scattering at low angles. In the next subsection we will give detailed results for non singlet and singlet components of electron structure function D.

2.1 The non singlet function

The electron structure function $D^e_e(x,\beta)$ is conveniently divided into 'valence' (NS) and 'sea' (S) parts, defined with the help of the $e \to \bar{e}$ function: $D^{NS} = D^e_e - D^{\bar{e}}_e$ and $D^S = D^{\bar{e}}_e$, so that

$$D^e_e = D^{NS} + D^S. \tag{19}$$

In the following discussion in this subsection we will discuss the nonsinglet component of eq. (19). The remaining part will be discused in the next subsection.

The iterative form of D^{NS} reads

$$D^{NS}(x,\beta) = \delta(1-x) + \sum_{k=1}^{\infty} \frac{1}{k!}\Big(\frac{\beta}{4}\Big)^k P^{\otimes k}(x), \quad P^{\otimes k}(x) = \underbrace{P\otimes\ldots\otimes P}_{k}(x), \tag{20}$$

$$P(x) = \delta(1-x)\Big(\frac{3}{2} + 2\ln\epsilon\Big) + \Theta(1-\epsilon-x)\,\frac{1+x^2}{1-x}, \tag{21}$$

$$P_1(\cdot)\otimes P_2(\cdot)(x) = \int_0^1 dx_1 dx_2 \delta(x - x_1x_2) P_1(x_1)P_2(x_2), \tag{22}$$

where $\epsilon \ll 1$ is an infrared regulator in the beam energy units, the limit $\epsilon \to 0$ is understood. The second and third order corrections to eq. (20) can be found in [10, 11] and [8] respectively. For the readers convenience we also collect them in Appendix A.

Another possible form of D^{NS}, regular and well behaved in IR limit $x \to 1$,results from applying to eq. (20) the exponentiation procedure of Ref. [12], see also Refs. [13, 14, 15] for more details. Including also third order corrections one finds [16, 17]

$$D^{NS}_{JW}(x,\beta) = D^{Gribov}(x,\beta)\Delta^{JW}(x,\beta), \tag{23}$$

$$D^{Gribov}(x,\beta) = \frac{\exp(\frac{1}{2}\beta(\frac{3}{4} - C_{Euler}))}{\Gamma(1+\frac{1}{2}\beta)}\frac{1}{2}\beta(1-x)^{\frac{1}{2}\beta - 1}. \tag{24}$$

$$\begin{aligned}\Delta^{JW}(x,\beta) \;=\; & \frac{1+x^2}{2} + \frac{1}{4}\frac{\beta}{2}\Big(-\frac{1}{2}(1+3x^2)\ln x - (1-x)^2\Big) + \frac{1}{8}(\frac{\beta}{2})^2\Big(\frac{1}{2}(3x^2-4x+1)\ln x \\ & + \frac{1}{12}(1+7x^2)\ln^2 x + (1-x^2)\mathrm{Li}_2(1-x) + (1-x)^2\Big) + \mathcal{O}(\beta^4).\end{aligned} \tag{25}$$

The convolution of two NS functions is done simply by duplication of the effective parameter β. We get therefore for the function I_2^{NS} of eq. (12)

$$I_2^{NS}(x,\beta) = D^{NS}(x, 2\beta) \tag{26}$$

and both forms (23) or (20) can be substituted here for D^{NS}. The integral I_1^{NS} of eq. (11) we have done up to second order without exponentiation. Since the IR points $x_1, x_2 = 1$ are excluded from I_1 we get

$$
\begin{aligned}
I_1^{NS}(x,a,\beta) &= \int\limits_{\sqrt{ax}}^{\sqrt{x/a}} \frac{dx_1}{x_1}\left(\frac{\beta}{4}\right)^2 \frac{1+x_1^2}{1-x_1}\frac{1+(\frac{x}{x_1})^2}{1-\frac{x}{x_1}} \\
&= \left(\frac{\beta}{4}\right)^2\left[2\sqrt{x}\left(\sqrt{a}-\frac{1}{\sqrt{a}}\right)+(x+1)\ln a+4\frac{1+x^2}{1-x}\ln\frac{1-\sqrt{ax}}{\sqrt{a}-\sqrt{x}}\right]. \qquad (27)
\end{aligned}
$$

Consequently

$$
\begin{aligned}
J_1^{NS}(a,\beta) &= \int\limits_{x_c}^{a} dx\frac{1}{x}I_1^{NS}(x,a,\beta) \\
&= 2\left(\frac{\beta}{2}\right)^2\Bigg[\mathrm{Li}_2(\sqrt{ax_c})-\mathrm{Li}_2\left(\sqrt{\frac{x_c}{a}}\right)+\mathrm{Li}_2(-\sqrt{a})+\mathrm{Li}_2(\sqrt{a})+\mathrm{Li}_2\left(\frac{\sqrt{a}-\sqrt{x_c}}{1-\sqrt{x_c}}\right) \\
&+ \mathrm{Li}_2\left(\frac{\sqrt{a}(1-\sqrt{x_c})}{1-\sqrt{ax_c}}\right)-\mathrm{Li}_2\left(\frac{\sqrt{a}(1+\sqrt{x_c})}{1+\sqrt{a}}\right)-\mathrm{Li}_2\left(\frac{\sqrt{a}-\sqrt{x_c}}{1+\sqrt{a}}\right)-\mathrm{Li}_2(a) \\
&- \frac{1}{2}\ln\sqrt{a}\ln\sqrt{\frac{a}{x_c}}+\frac{\pi^2}{6}+\ln(1-\sqrt{a})\ln\left(\frac{\sqrt{a}-\sqrt{x_c}}{1-\sqrt{ax_c}}\right)+\frac{1}{2}\ln^2(1-\sqrt{ax_c}) \\
&- \ln(1+\sqrt{a})\ln\left(\frac{1+\sqrt{a}}{1+\sqrt{x_c}}\right)+\frac{1}{2}\ln^2(1-\sqrt{x_c})+\ln(\sqrt{a}-\sqrt{x_c})\ln\left(\frac{1+\sqrt{a}}{1+\sqrt{x_c}}\right) \\
&- \ln(\sqrt{a}-\sqrt{x_c})\ln(1-\sqrt{x_c})+\frac{1}{8}(a-x_c)\ln a-\frac{1}{2}\left(a-\frac{1}{a}\right)\ln(1-a) \\
&+ \frac{1}{2}\left(x_c-\frac{1}{a}\right)\ln(1-\sqrt{ax_c})-\frac{1}{2}(x_c-a)\ln(\sqrt{a}-\sqrt{x_c})\Bigg]. \qquad (28)
\end{aligned}
$$

The dilogarithm function is defined in a usual way: $\mathrm{Li}_2(a)=-\int_0^1(\ln(1-ax)/x)dx$.[4]

2.2 The singlet function

Up to the $\mathcal{O}(\beta^3)$ the singlet function reads [8]

$$
\begin{aligned}
D^S(x,\beta) &= \frac{1}{2!}\beta^2\frac{1}{16}R(x)+\frac{1}{3!}\beta^3\left(\frac{1}{32}P\otimes R(x)-\frac{1}{96}R(x)\right), \qquad (29)\\
R(x) &= \frac{1-x}{3x}(4+7x+4x^2)+2(1+x)\ln x, \qquad (30)\\
P\otimes R(x) &= \left(\frac{3}{2}+2\ln(1-x)\right)R(x)+(1+x)(-\ln^2 x+4\mathrm{Li}_2(1-x)) \\
&+\frac{1}{3}(-9-3x+8x^2)\ln x+\frac{2}{3}\left(-\frac{3}{x}-8+8x+3x^2\right). \qquad (31)
\end{aligned}
$$

[4]Note that $J_1^{NS}(a=1)=(\beta^2/2)(\pi^2/4)$. It seems to be in apparent contradiction to eq. (27), which states that $I_1^{NS}(a=1)=0$. However, the interchange of limit $a\to 1$ and integration in eq. (28) is not legitimate because the integrand is not uniformly convergent.

According to the definition (19) the singlet component of I_2 is the following

$$
\begin{aligned}
I_2^S(x,\beta) &= \int\limits_x^1 \frac{dx_1}{x_1}\Big[D^S(x_1)D^S\Big(\frac{x}{x_1}\Big) + 2D^S(x_1)D^{NS}\Big(\frac{x}{x_1}\Big)\Big] \\
&= \Big(\frac{\beta}{4}\Big)^2 R(x) + \Big(\frac{\beta}{4}\Big)^3\Big(\frac{5}{3}P\otimes R(x) - \frac{2}{9}R(x)\Big) + \mathcal{O}(\beta^4). \qquad (32)
\end{aligned}
$$

In a similar manner the I_1^S function up to $\mathcal{O}(\beta^4)$ reads

$$
\begin{aligned}
I_1^S(x,a,\beta) &= \int\limits_{\sqrt{ax}}^{\sqrt{x/a}} \frac{dx_1}{x_1}\Big[D^S(x_1)D^S\Big(\frac{x}{x_1}\Big) + 2D^S(x_1)D^{NS}\Big(\frac{x}{x_1}\Big)\Big] \\
&= \Big(\frac{\beta}{4}\Big)^3 \int\limits_{\sqrt{ax}}^{\sqrt{x/a}} \frac{dx_1}{x_1} R(x_1)\frac{1+(\frac{x}{x_1})^2}{1-\frac{x}{x_1}} \\
&= \Big(\frac{\beta}{4}\Big)^3\Big[\frac{2}{3}\Big(a-\frac{1}{a}\Big)(1+x) + \Big(\sqrt{a}-\frac{1}{\sqrt{a}}\Big)\Big(6\sqrt{x}+\frac{4}{3}\frac{1}{\sqrt{x}}+\frac{4}{3}x\sqrt{x}\Big) \\
&+ \frac{4}{3}\Big(\frac{1}{x}+x^2\Big)\ln a - 2\sqrt{x}\Big(\sqrt{a}+\frac{1}{\sqrt{a}}\Big)\ln a + (1+x)\ln a\ln x \\
&+ \frac{2}{3}\Big(\frac{4}{x}+3-3x-4x^2\Big)\ln\frac{1-\sqrt{ax}}{\sqrt{a}-\sqrt{x}} + 4(1+x)\int\limits_{\sqrt{ax}}^{\sqrt{x/a}} dx_1\frac{\ln x_1}{x_1-x}\Big]. \qquad (33)
\end{aligned}
$$

As in the NS case, in the above we benefited from the fact that IR points are excluded from the integration domain and also that the expansion of D^S starts at second order in β. The integral in the last line of eq. (33) can be alternatively expressed in terms of dilogarithm functions. We keep this integral form for the purpose of further integration over x first. Finally, we get

$$
\begin{aligned}
J_1^S(a,\beta) &= \Big(\frac{\beta}{4}\Big)^3\Big\{(\sqrt{a}-\sqrt{x_c})\Big[\frac{4}{3}(a^{-\frac{3}{2}}-a^{\frac{3}{2}}) + 2\Big(\sqrt{a}-\frac{1}{\sqrt{a}}\Big)\Big] \\
&+ \frac{4}{9}(a^{\frac{3}{2}}-x_c^{\frac{3}{2}})\Big(\sqrt{a}-\frac{1}{\sqrt{a}}\Big) + \Big[\frac{4}{3}\Big(\frac{1}{x_c}-\frac{1}{a}\Big) + \frac{2}{3}(a^2-x_c^2) + \frac{2}{3}\Big(\frac{1}{a}-a\Big) + a - x_c \\
&\quad -4 + 2\sqrt{\frac{x_c}{a}} + 2\sqrt{ax_c} - (1+x_c)\ln a + (1+a)\ln x_c + \frac{1}{2}\ln^2 x_c - \frac{1}{2}\ln^2 a\Big]\ln a \\
&+ 2\Big[-\frac{1}{3}\Big(\frac{1}{a}-a\Big) + \sqrt{\frac{x_c}{a}} - \sqrt{ax_c}\Big]\ln x_c + \Big[\frac{4}{3}\Big(\frac{1}{a^2}-a^2\Big) + \frac{10}{3}\Big(\frac{1}{a}-a\Big)\Big]\ln(1-a) \\
&+ \Big[\frac{8}{3}\Big(\frac{1}{x_c}-a\Big) + \frac{4}{3}\Big(x_c^2-\frac{1}{a^2}\Big) + 6\Big(x_c-\frac{1}{a}\Big) - 2\Big(x_c-\frac{1}{a}\Big)\ln\frac{x_c}{a}\Big]\ln(1-\sqrt{ax_c}) \\
&+ \Big[\frac{8}{3}\Big(\frac{1}{a}-\frac{1}{x_c}\Big) + \frac{4}{3}\Big(a^2-x_c^2\Big) + 6\Big(a-x_c\Big) + 2\Big(x_c-a\Big)\ln(ax_c)\Big]\ln(\sqrt{a}-\sqrt{x_c}) \\
&+ 4\Big[1+2(a+\ln a)\Big]\mathrm{Li}_2(1) - 4\Big[1+a+\frac{1}{a}\Big]\mathrm{Li}_2(a)
\end{aligned}
$$

$$+ \quad 4\Big[1+\frac{1}{a}+x_c+\ln\frac{x_c}{a}\Big]\mathrm{Li}_2(\sqrt{ax_c}) - 4\Big[1+a+x_c+\ln(ax_c)\Big]\mathrm{Li}_2\Big(\sqrt{\frac{x_c}{a}}\Big)\Big\}. \tag{34}$$

This completes the determination of the analytical LL formulae for pair production in the LEP/SLC luminosity process to second order in the NS case and third order in the S case. In the next section, we illustrate these formulae in the LEP/SLC luminosity regime.

3 Results and Applications

In this section we evaluate our formulae for the effects of pair production on the LEP/SLC luminosity process for the narrow, mixed and wide type cuts defined in Refs. [1, 6]. We also present a recipe for including this pair production effect into the luminosity analysis using our existing Monte Carlo program BHLUMI 2.01 [1].

Specifically, we begin by discussing the size of the NS pair effect in our prototypical narrow (N), mixed (M), and wide (W) triggers as they are defined in Ref. [6, 1]. This is shown in Fig. 4, as a function of x_{cut}, the minimum value of the trigger calorimetric invariant squared mass of the outgoing charged particles into the trigger acceptance. The plotted cross section results from numerical integration of eq. (14) with the third order exponentiated function I_2^{NS} of eqs. (23)–(25) and the second order form (28) of the J_1^{NS} function. The Fig. 4 is a difference of the above cross section with running parameter β_R of eq. (3) and nonrunning β of eq. (2), normalized to the Born cross section, at $\sqrt{s} = M_Z$. We see that the value of the pair effect, σ^{NS}_{pair}, in the typical region $x_{cut} \leq 0.7$ is entirely consistent with the estimates in Table 3 of the first paper in Ref. [1]. The technical accuracy of the integration in eq. (14) we estimate conservatively to be at least 7-8 digits. It is done by comparison of two independent integration programs of Monte Carlo and Gaussian type. The physical accuracy of σ^{NS}_{pair} is limited by unincluded higher order and nonleading terms. The $\mathcal{O}(\beta^4)$ correction to σ^{NS}_{pair} coming from exponentiated function I_2^{NS} is negligible. This can be seen by inspection of the $\mathcal{O}(\beta^3)$ correction which is already of order 10^{-3} with respect to the first order term. The J_1^{NS} function, beginning at order β^2, and calculated only to this order, contributes, in the worst case, for loose cuts up to 1/3 of the $\mathcal{O}(\beta^1)$ pair effect. We assume, therefore, the corresponding higher order error $\Delta\sigma^{NS}_{pair}$ to be less than 15% of σ^{NS}_{pair}, or more specifically 40% of the actual J_1 contribution. The nonleading terms we estimate, in analogy to the s–channel results of Ref. [4], to be no bigger than 30% of the leading terms. Altogether , we conservatively estimate the physical uncertainty $\Delta\sigma^{NS}_{pair}$ to be less than 45% of its actual size.

We continue our discussion of our results and applications by showing the size of the singlet contribution to the pair effect in our prototypical LEP/SLC trigger. This we do by plotting in Fig. 5 the third order singlet cross section resulting from numerical integration of eq. (14) with functions (32) and (34) as a function of x_{cut} at $\sqrt{s} = M_Z$, normalized to the NS Born cross section. Again, the size of the singlet contribution is consistent with the upper bound shown in Table 3 of the first paper in Ref. [1]. Similarly to the NS case, the technical precision of our integration, as determined by comparison of two independent programs, we estimate to be at least 10 digits. The physical accuracy of σ^{S}_{pair} is limited by omitted $\mathcal{O}(\beta^4)$ and nonleading terms. The $\mathcal{O}(\beta^4)$ terms we estimate by inspecting the size of $\mathcal{O}(\beta^3)$ terms. Since $\Delta\sigma^{S,3ord}_{pair} < 0.2 \cdot \Delta\sigma^{S,2ord}_{pair}$ we estimate $\Delta\sigma^{S,4ord}_{pair}$ to be less than 5% of σ^{S}_{pair}. Keeping also the 30% limit for nonleading terms we finally get $\Delta\sigma^{S}_{pair}$ to be less than 35% of the σ^{S}_{pair}.

Finally, we illustrate the energy dependence of the pair effect in Table 1. We use the singlet third order cross section for the purpose of this illustration but an entirely analogous dependence obtains in the nonsinglet case. We see that, for each type of trigger, the main effect in the change of $\sqrt{s}$ from M_Z to $M_Z \pm \Gamma/2$ is the $\frac{1}{s}$-dependence of the cross section. Of course, there is more to the $s-$dependence of the pair effect than just this scale invariance factor but it is refreshing to see that it dominates the $s-$dependence near the Z^0 resonance.

The results in Figs. 4 and 5 and Table 1 were obtained by implementing the formulae in the previous section in a program BHPAIR which is available from the authors upon request.

Evidently, the program BHPAIR can be used to correct the luminosity analyses of LEP/SLC experiments for the pair effect in a way analogous to that in which LUMLOG was used to correct the LEP luminosity analyses according to the recipe in Ref. [6]. One simply runs the BHPAIR program for the same cuts as the data and thereby determines the size of the pair effect on the theoretical cross section.This can then be added to the expected cross section from BHLUMI2.01,for example. A more desirable implementation would be to have the extra pair events produced with their proper physical 4-vectors at the level of the BHLUMI Monte

Carlo so that the pairs could be sent through the detector simulation as well. This would overcome the collinear LL ansatz inherent in the earlier correction prescription for example. Such an implementation is already in its prototype stage and will appear in its complete form elsewhere [7].

4 Conclusions

In this paper we have presented new analytical formulae for the cross section for the pair production effect in low angle Bhabha scattering in the LEP/SLC luminosity regime. We have in this way removed one of the contributions to the uncertainty in the theoretical prediction for this luminosity at the .01% level. In addition, we have corroborated the initial error estimates in Ref. [1, 6] for this effect. Thus, we have taken an important step in the reduction of the theoretical error on the LEP/SLC luminosity process to the .05% regime. Further such steps will appear elsewhere [7].

Acknowledgments

Two of us (S.J. and B.F.L.W) thank Profs. G. Feldman of Harvard, J. Dorfan of SLAC and M. Breidenbach of SLAC for allowing them to participate in the MKII and SLD SLC Radiative Corrections Working Groups wherein the genesis for this research was formed. These authors further thank Prof. J. Ellis and the ALEPH Collaboration for the support and kind hospitality of the CERN TH and PPE Divisions where this research was completed.

Appendix A

The second order correction to eq. (20) reads [10, 11]

$$\frac{1}{2}P\otimes P(x) = \delta(1-x)\Big(2\ln^2\epsilon + 3\ln\epsilon + \frac{9}{8} - 2\zeta(2)\Big) + \Theta(1-\epsilon-x)$$
$$\Big[\frac{1+x^2}{1-x}(2\ln(1-x) - \ln x + \frac{3}{2}) + \frac{1}{2}(1+x)\ln x - 1 + x\Big]. \qquad (35)$$

The third order term reads [8]

$$\frac{1}{3!}\left(\frac{\beta}{4}\right)^3 P\otimes P\otimes P(x)=\left(\frac{\beta}{2}\right)^3\delta(1-x)\delta_{term}+\left(\frac{\beta}{2}\right)^3\Theta(1-\epsilon-x)\theta_{term} \tag{36}$$

with

$$\begin{aligned}
\theta_{term} &= \Big[\frac{1}{2}\frac{1+x^2}{1-x}\Big(\frac{9}{32}-\frac{\pi^2}{12}+\frac{3}{4}\ln(1-x)-\frac{3}{8}\ln x+\frac{1}{2}\ln^2(1-x)\\
&\quad+\frac{1}{12}\ln^2 x-\frac{1}{2}\ln x\ln(1-x)\Big)+\frac{1}{8}(1+x)\ln x\ln(1-x)\\
&\quad-\frac{1}{4}(1-x)\ln(1-x)+\frac{1}{32}(5-3x)\ln x-\frac{1}{16}(1-x)\\
&\quad-\frac{1}{32}(1+x)\ln^2 x+\frac{1}{8}(1+x)\mathrm{Li}_2(1-x)\Big], && (37)\\
\delta_{term} &= \frac{1}{3}\zeta(3)-\frac{1}{2}\zeta(2)\Big(\frac{3}{4}+\ln\epsilon\Big)+\frac{1}{6}\Big(\frac{3}{4}+\ln\epsilon\Big)^3 && (38)
\end{aligned}$$

where $\zeta(2)=\pi^2/6$ and $\zeta(3)=1.2020569031\ldots$.

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Figure Captions

1. Integration domain in $x_1 - x_2$ space for our cross section.The curve $x = a$ separates the integration region into a two real photon/pair emission region $(x < a)$ and the remaining region $(x \geq a)$ which contains the infrared points.

2. The same integration regime as in Fig. 1 in the $x - z$ space as defined in the text.

3. The asymmetric cuts on our cross section represented as a sum of symmetric cut cross sections with appropriate weights as given in eq. (18).

4. Effect of NS pairs for our three generic cuts, $NN \equiv N$, $WW \equiv W$ and $\frac{1}{2}(NW + WN) \equiv M$. The curves represent difference of the cross section (14) with third order exponentiated function I_2^{NS} of eqs. (23)–(25) and second order form (28) of J_1^{NS} with running β_R of eq. (3) minus the similar cross section with nonrunning β of eq. (2), normalized to the Born cross section, at $\sqrt{s} = M_Z$.

5. Effect of singlet pairs relative to the NS Born cross section for the three generic cuts N,W and M defined in Fig. 4. The curves are third order singlet cross sections resulting from numerical integration of eq. (14) with functions (32) and (34) as a functions of x_{cut} at $\sqrt{s} = M_Z$.

Table Caption

1. Comparison of the third order singlet contribution for pairs at different values of $\sqrt{s}$ for our generic cuts W,N and M ,for representative values of x_{cut}. We see that the main energy dependence is just the expected $\frac{1}{s}$ behavior.

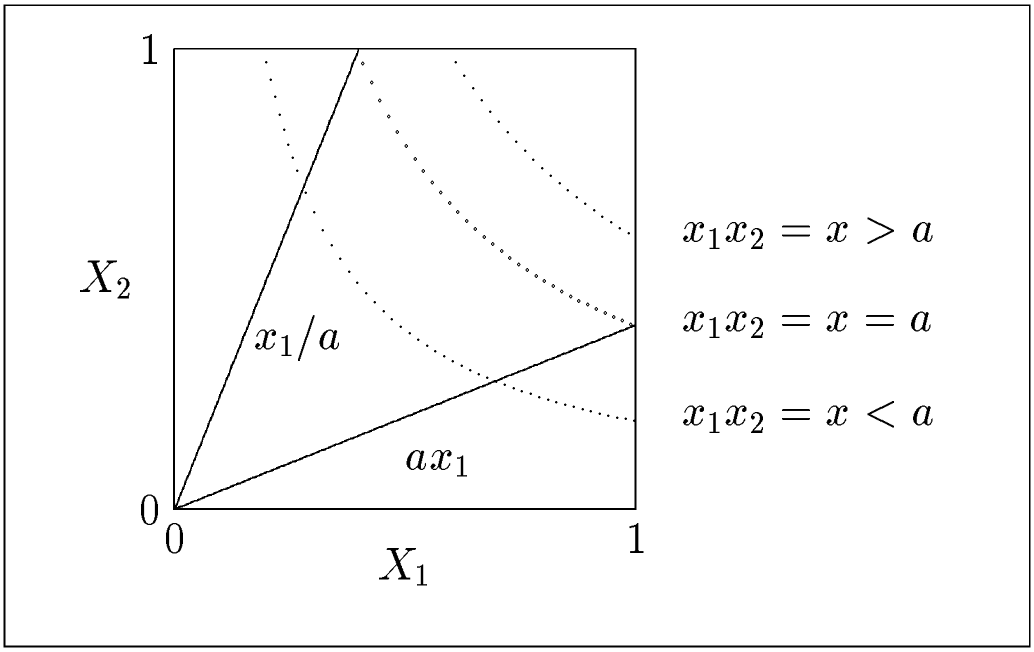
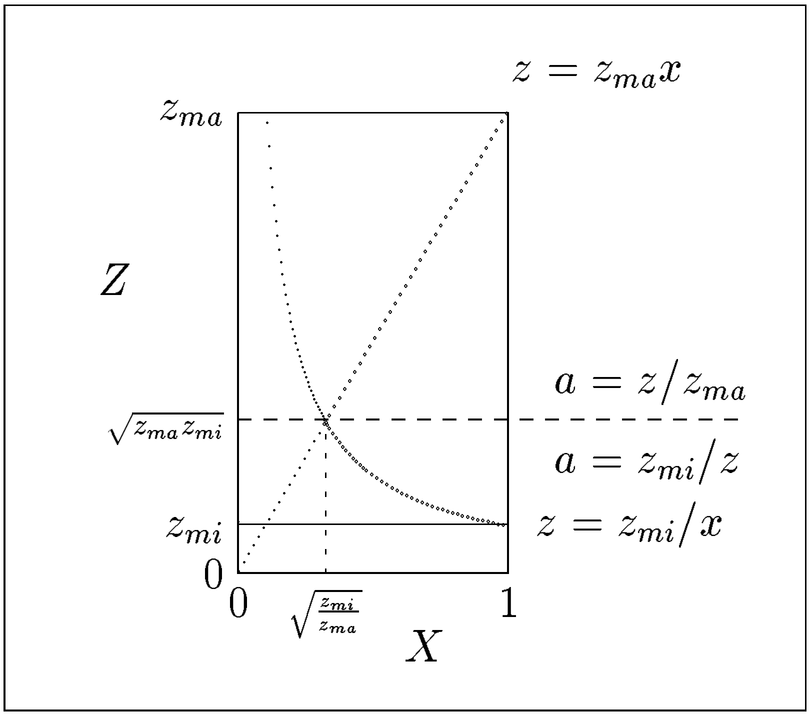


Fig. 1



$z = z_{ma}x$
z_{ma}
Z
$a = z/z_{ma}$
$\sqrt{z_{ma}z_{mi}}$
$a = z_{mi}/z$
z_{mi}
$z = z_{mi}/x$
0
0
$\sqrt{\frac{z_{mi}}{z_{ma}}}$
1
X

Fig. 2

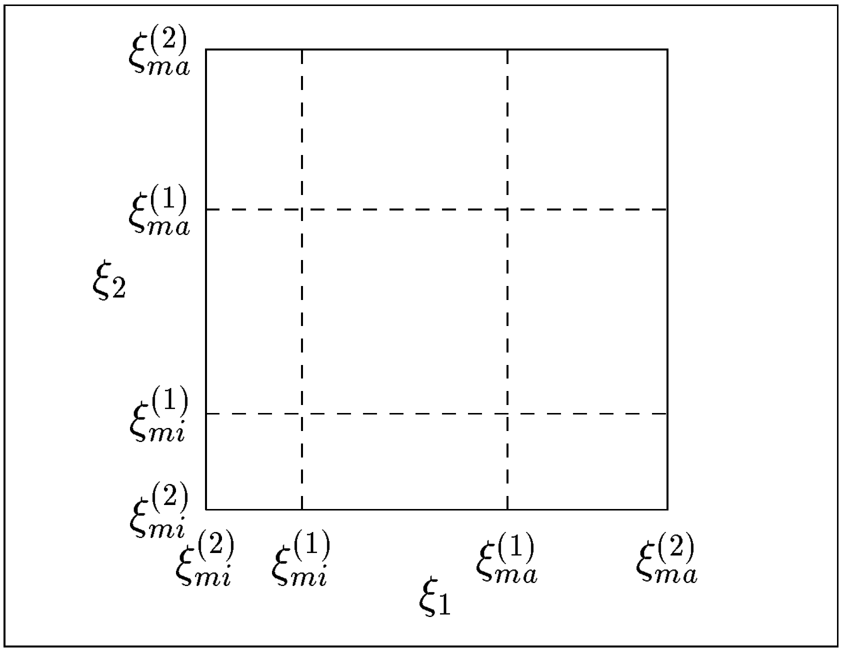


Fig. 3

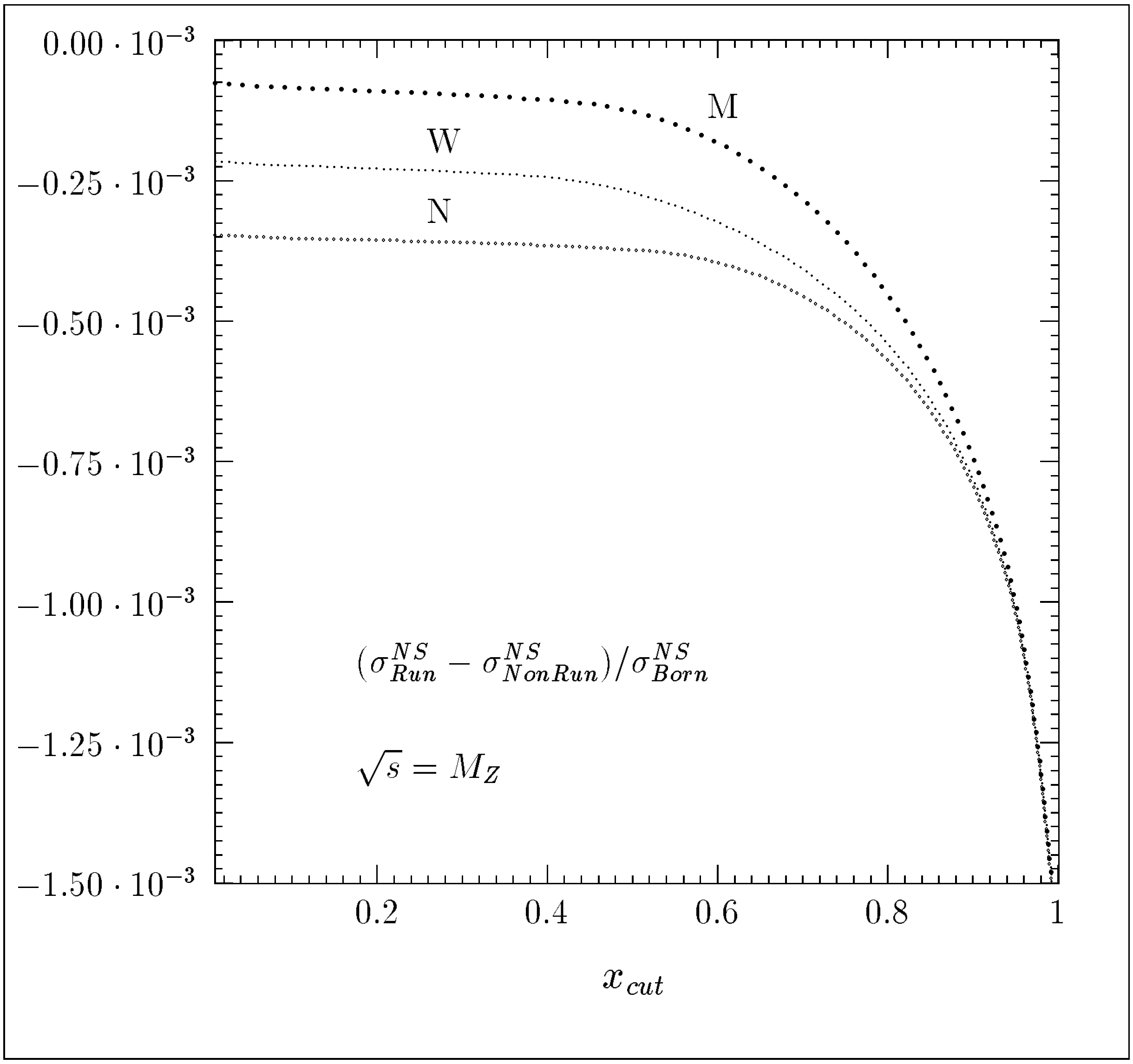

0.00 · 10^-3
−0.25 · 10^-3
−0.50 · 10^-3
−0.75 · 10^-3
−1.00 · 10^-3
−1.25 · 10^-3
−1.50 · 10^-3
0.2
0.4
0.6
0.8
1
x_{cut}
M
W
N
$(\sigma^{NS}_{Run} - \sigma^{NS}_{NonRun})/\sigma^{NS}_{Born}$
$\sqrt{s} = M_Z$

Fig. 4

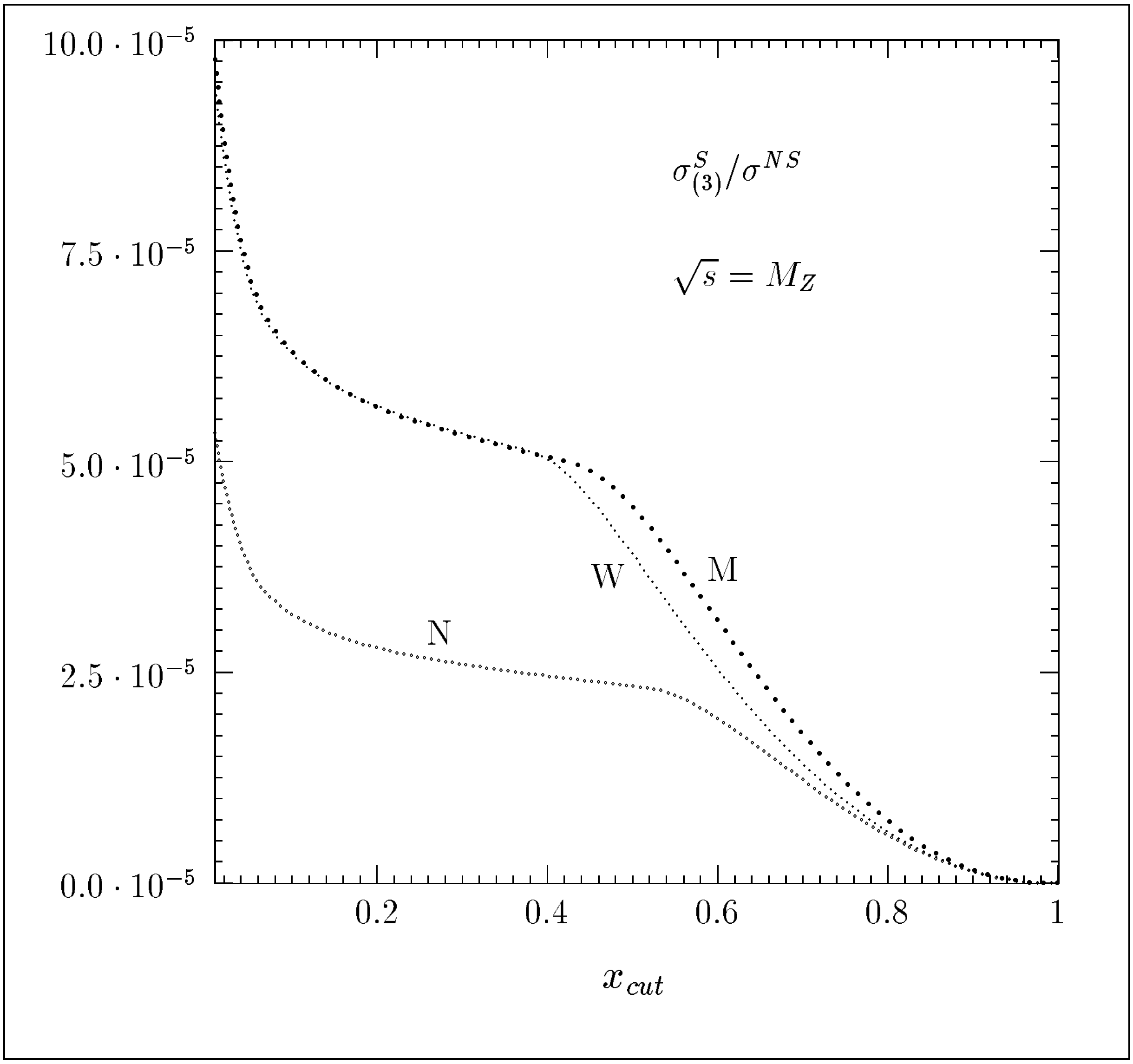

$10.0 \cdot 10^{-5}$
$7.5 \cdot 10^{-5}$
$5.0 \cdot 10^{-5}$
$2.5 \cdot 10^{-5}$
$0.0 \cdot 10^{-5}$
0.2
0.4
0.6
0.8
1
x_{cut}
$\sigma^{S}_{(3)}/\sigma^{NS}$
$\sqrt{s} = M_Z$
W
M
N

Fig. 5

x_{cut}	Triger	$\sigma^S_{(3)}(M_Z+\Gamma/2)/\sigma^S_{(3)}(M_Z)$	$\sigma^S_{(3)}(M_Z-\Gamma/2)/\sigma^S_{(3)}(M_Z)$
0.2	W	0.976871	1.023995
	M	0.976893	1.023971
	N	0.976962	1.023898
0.5	W	0.976642	1.024238
	M	0.976650	1.024229
	N	0.976699	1.024178
0.8	W	0.976477	1.024413
	M	0.976467	1.024424
	N	0.976482	1.024408

Table 1