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# Higher order QED corrections to Bhabha scattering at low angles*

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**Abstract**

We calculate the QED bremsstrahlung corrections beyond $O(\alpha)$ for small angle Bhabha scattering. These corrections are of vital importance for measuring the luminosity and consequently the total cross sections in LEP experiments. They are dominated by the second order leading-logarithmic contributions. We calculate them with a high *technical* precision 0.02% and we show that they depend strongly on details of experimental cut-offs. We propose a practical method of introducing $O(\alpha^2)$ correction calculated in this paper into current procedures of the luminosity measurement at LEP. The remaining overall QED uncertainty is estimated to be 0.3%. This estimate includes 0.2% due to pure bremsstrahlung and 0.1% due to light pair production but does not include hadronic uncertainty in the photon vacuum polarization.

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Since the beginning of their operation all LEP experiments accumulated a big amount of experimental data and in addition a substantial increase of statistics is expected in 1991. With the decreasing statistical errors, systematic errors become more and more important. Generally, there are two main sources of the systematic errors in the experimental results: the purely experimental one, which is determined by the quality of the detector and accelerator, the level of understanding their operation, and the theoretical one which *necessarily* enters into systematic experimental error if the experimental quantity depends substantially on some theoretical calculation. This typically happens in the case of background subtraction, model dependent parametrization of the data, luminosity measurement, etc.

In the luminosity measurement it is necessary to measure very accurately at least one scattering process for which the total cross section is theoretically known and calculable with sufficiently high precision. In $e^+e^-$ machines the small angle Bhabha cross section is used for this purpose. Even though, in principle, small angle Bhabha scattering can be calculated in QED with arbitrarily high precision (for the $t$-channel photon exchange part at least), so far, only results at the precision level of 1% are available (mostly from PETRA/PEP times) [1]. It does not match the actual experimental precision, which is now approaching 0.6% level. At present, the systematic error related to theoretical uncertainty in the calculations of the luminosity cross section is among dominant factors limiting the usefulness of many LEP results [2].

The most convenient and reliable calculation of the theoretical luminosity cross section within experimental cuts is done with the Monte Carlo (M.C.) numerical calculation. At present, the existing Monte Carlo event generators for small angle Bhabha are based on the $O(\alpha)$ QED calculations only: OLDBAB [3], BABAMC [4], while BHLUMI [5] includes part of the higher order QED corrections due to exclusive exponentiation in the Yennie-Frautschi-Suura [6] style. Most of LEP collaborations determine presently luminosity using the integrated $O(\alpha)$ cross section from the BABAMC program. The comparison of results from these M.C. programs suggests that they provide a theoretical small angle Bhabha cross-section with an overall precision of the of order 1%. Recently, an effort was started to improve substantially on this precision. In ref. [7] the first elementary step in this direction has been done. A method of determining *the technical precision* of any $O(\alpha)$ M.C. was given. By technical precision we understand all kinds of errors due to numerical approximations and mistakes in the analytical/numerical calculations, including rounding errors, programming bugs, random number effects, etc., but not higher order effects new physics, etc. The results of ref. [7], programs and tables, form a *numerical benchmark* which allows to determine[1] whether a given $O(\alpha)$ M.C. provides the true QED $O(\alpha)$ cross section within a technical precision of 0.02%.

[1]The BABAMC versions used by ALEPH, OPAL and DELPHI were recently validated [8, 9, 10] with help of the numerical benchmark of ref. [7] in the small angle regime below the 0.1% precision level. The only bias due to the so-called $k_0$ parameter was found. It is enough to set $k_0 = 10^{-3}$ to keep it below 0.1%.

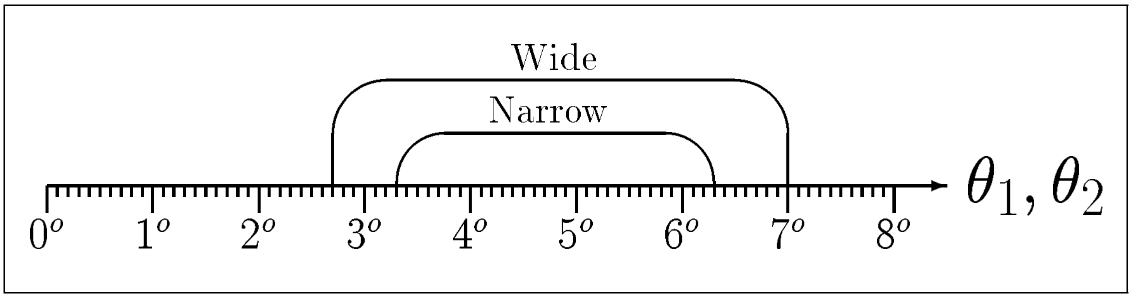


Figure 1: *Three types of the angular trigger used in numerical calculations in this paper: (1) asymmetric, very close to typical luminosity LEP trigger, denoted as* Narrow-Wide (N-W) *and defined by* $\vartheta^N_{min} < \vartheta_1 < \vartheta^N_{max}$ *and* $\vartheta^W_{min} < \vartheta_2 < \vartheta^W_{max}$ *and two symmetric, less realistic but easier for semi-analytical calculations, (2)* Narrow-Narrow (N-N) $\vartheta^N_{min} < \vartheta_{1,2} < \vartheta^N_{max}$ *and (3)* Wide-Wide (W-W) $\vartheta^W_{min} < \vartheta_{1,2} < \vartheta^W_{max}$. *We shall use:* $\vartheta^W_{min} = 2.7^o$, $\vartheta^W_{max} = 7^o$, $\vartheta^N_{min} = 3.3^o$ *and* $\vartheta^N_{max} = 6.3^o$. *We define* $\vartheta_i$ *as an angle between scattered* $e^\pm$ *and the corresponding (equal sign) beam* $e^\pm$.

The aim of this work is to calculate the largest higher order QED bremsstrahlung correction beyond the $O(\alpha)$, i.e., *initial state* $O(\alpha^2)$ *leading-logarithmic (LL) contribution with the same technical precision of 0.02% and for true experimental cuts.* We shall provide a practical method of calculating this correction with help of the simple and well tested LL M.C. event generator and describe how to include it into current experimental luminosity measurements. We shall also estimate/calculate the size of the other higher order QED corrections (second order subleading and pairs) which are left out and still enter into the systematic error.

It should be stressed that the above method of reducing theoretical uncertainty in the luminosity measurement is a *temporary solution* and the real solution is to develop and test the $O(\alpha^2)$ M.C. event generator for the small angle Bhabha process. The best candidate for such a M.C. is the multi-photon M.C. event generator BHLUMI [5] which already integrates over multi-photon hard photon phase space. It requires improvement of the matrix element and solid tests of its technical precision[2], without any major change of the M.C. generation algorithm.

Let us note that there exists also a semi-analytical calculation, aimed mainly into wide angle Bhabha scattering [11] which at low angles has, according to the authors, 0.5% physical precision [12]. It could be useful in a test of the M.C. event generators, for simplified cut-offs, if its technical/numerical precision was known or specified.

Before we come to any numerical calculations let us define semi-realistic cuts used in this paper. We shall use one asymmetric and two symmetric types of the angular trigger defined in Fig. 1. The additional energy cut will be imposed on variable $s'/s$ where $\sqrt{s'}$ is an effective mass of the final $e^\pm$ pair[3].

[2]This work is under development and its results will be published elsewhere.

[3]We shall always define explicitly whether we understand the final state $e^\pm$ as calorimetric

Since the main calculations of this paper will be done in the leading-logarithmic (LL) approximation let us therefore ask a basic question for the study of systematic uncertainties: *how big are subleading (next-to-leading) contributions with respect to leading?* This question can be answered quite precisely in the O($\alpha$) because, since the work of ref. [7], the corresponding O($\alpha$) calculation is feasible with 0.02% technical/numerical precision. In the decomposition

$$\sigma^{O(\alpha)} = \sigma^{O(\alpha)}_{LL} + \sigma^{O(\alpha)}_{subl} \tag{1}$$

let us define explicitly the LL O($\alpha$) component as a respective truncation of the analytical formula in ref. [7]

$$\begin{aligned}
\sigma^{O(\alpha)}_{LL} = \frac{2\pi\alpha^2}{s}\int\limits_{\xi_{min}}^{\xi_{max}} \frac{d\xi}{\xi^2}\Bigg[ & W(\xi)\,(1+4\beta_t\ln\epsilon+3\beta_t)\\
& + \int\limits_{\epsilon}^{k_{max}} \frac{dk}{k}\, W(k)\,\beta_t\, W\left(\frac{\xi}{1-k+k\xi}\right)\Theta^{\xi_{max}}_{\xi_{min}}\left(\frac{\xi}{1-k(2-k)(1-\xi)}\right)\\
& + \int\limits_{\epsilon}^{k_{max}} \frac{dk}{k}\, W(k)\,\beta_t\, \frac{1}{(1-k)^2} W\left(\frac{(1-k)\xi}{1-k\xi}\right)\Theta^{\xi_{max}}_{\xi_{min}}\left(\frac{(1-k)^2\xi}{1-\xi k(2-k)}\right)\\
& + \int\limits_{\epsilon}^{k_{max}} \frac{dk}{k}\, W(k)\,\beta_t\, W(\xi) + \int\limits_{\epsilon}^{k_{max}} \frac{dk}{k}\, W(k)\,\beta_t\, \frac{1}{(1-k)^2}\, W(\xi)\Bigg],
\end{aligned} \tag{2}$$

where $k$ denotes the energy of the photon emitted from initial or final state in units of the beam energy, $k_{max}$ defines maximum photon energy and $\epsilon = k_0$ is the soft photon regulator. Following ref. [7] we translate the scattering angle $\vartheta$ into $\xi = (1-\cos\vartheta)/2$ and the following short-hand notation is used $W(x) = (1+(1-x)^2)/2$ and $\Theta^{\xi_{max}}_{\xi_{min}}(\xi) = \Theta(\xi-\xi_{min})\Theta(\xi_{max}-\xi)$. In the above expression the symmetric angular trigger, either N-N or W-W trigger of Fig. 1, is understood. In the LL approximation there is always freedom in the definition of the big logarithm parameter $\beta_t$. We shall compare results for the following two definitions

$$\beta_t^{(A)} = \frac{\alpha}{\pi}\left(\ln\frac{|t|}{m_e^2}-1\right),\quad \beta_t^{(B)} = \frac{\alpha}{\pi}\ln\frac{s\xi_{min}}{m_e^2} = \frac{\alpha}{\pi}\ln\frac{|t_{min}|}{m_e^2}. \tag{3}$$

In the case of $\beta_t^{(A)}$ the $|t|$ denotes the true four-momentum transfer exchanged in the hard scattering between the two electron lines. This implies that $\beta_t^{(A)}$ in eq. (3) is a function[4] of $k$ and $\xi$. The $\beta_t^{(A)}$ choice sounds theoretically better because it has the proper soft photon limit, i.e., in the soft limit $k_{max} \to 0$ the corresponding

clusters or the real $e^\pm$.

[4]The four $k$-integrals in eq. (3) represent bremsstrahlung from four initial/final state $e^\pm$. In the first integral $|t| = s(1-k)\xi/(1-k+k\xi)$, in the second one $|t| = s\xi(1-k)^2/(1-k\xi)$ and in the remaining two $|t| = s\xi(1-k)$.

| *trigger* | $k_{max}$ | $\delta^{O(\alpha)}$ | $\delta^{O(\alpha)}_{subl,(A)}$ | $\delta^{O(\alpha)}_{subl,(B)}$ |
|---|---|---|---|---|
| Narrow-Narrow | 0.9999 | -7.931% | -1.191% | -1.064% |
| | 0.5 | -10.328% | -0.626% | -0.462% |
| | 0.1 | -26.434% | -0.332% | -0.407% |
| Wide-Wide | 0.9999 | -5.351% | -0.926% | -0.907% |
| | 0.5 | -8.115% | -0.541% | -0.471% |
| | 0.1 | -25.702% | -0.306% | +0.192% |

Table 1: *The split of the $O(\alpha)$ correction into LL and subleading contributions at $\sqrt{s} = 92 GeV$, for the two angular triggers of Fig. 1 and for two definitions of LL parameter $\beta_t$. The $k_{max}$ cut limits sum of the energies of the real electrons (not calorimetric!). Subleading correction is calculated as a difference between $O(\alpha)$ result of ref. [7] and the $O(\alpha)$ LL contribution as defined in eq. (3). All results have numerical/technical precision $\delta\sigma/\sigma = 0.02\%$. Vacuum polarization and s-channel exchange are excluded.*

subleading correction tends to a known constant. For $\beta_t^{(B)}$ the corresponding subleading correction diverges in the soft limit[5] (the same is true for the $\beta_t$ choice of ref. [12]).

In Table 1 we show the O($\alpha$) subleading correction defined as $\delta^{O(\alpha)}_{subl} = (\sigma^{O(\alpha)} - \sigma^{O(\alpha)}_{LL})/\sigma_{Born}$ and calculated using formulas and programs of ref. [7]. They indicate that subleading corrections are smaller than the total O($\alpha$) LL correction by the healthy ratio one to five. This result is true for two types of symmetric angular cut, for various strengths of the energy cut and two types of the big-log definition. *We may therefore trust the LL approximation for small angle Bhabha.* This exercise should be repeated as an important cross-check for the true (asymmetric) experimental cuts of each experiment and we shall explain later in the paper how to do it in practice. Let us note for the purpose of our final discussion that for a typical energy cut $k_{max} \approx 0.5$, the value of subleading correction depends only slightly on the definitions of $\beta_t$, i.e., it changes by $\approx 0.2\%$.

Having gained confidence into the LL approximation, let us now define the framework of our LL calculations. Contrary to the above O($\alpha$) exercise in the following LL calculations we shall neglect the LL final state bremsstrahlung because in the

[5]This choice is perfectly legitimate for LL calculation, the divergence illustrates only the well known fact that the LL approximation has generally incorrect soft photon behavior.

standard luminosity measurement it is equal to zero. In the typical luminosity measurement the final electron and the collinear photons is seen in the calorimeter as a single object (dressed electron). In this case the LL final state contribution cancels out in any perturbative order due to Lee-Kinoshita-Nauenberg theorem [13]. Traces of information on the details of the procedure of combining electron and photon into dressed electron (calorimeter granulation, limiting angle for seeing electron and photon separately, edge effects etc.) survive in the non-logarithmic (subleading) correction[6] only. The O($\alpha$) subleading corrections are necessarily included in the corresponding M.C. calculation.

The master formula for the small angle Bhabha total cross section with the initial state bremsstrahlung in the LL approximation reads

$$\begin{aligned}\sigma_{LL} = \quad & \int\limits_0^1 dx_1 \int\limits_0^1 dx_2 D(x_1,\beta_t)D(x_2,\beta_t) \int d\xi^* \\ & \frac{d\sigma^{Born}}{d\xi^*}\ \Theta_{\xi^{(1)}_{min}}^{\xi^{(1)}_{max}}(\xi_1)\ \Theta_{\xi^{(2)}_{min}}^{\xi^{(2)}_{max}}(\xi_2)\ \Theta(v_{max}-1+x_1x_2),\end{aligned} \tag{4}$$

where $x_1, x_2$ denote fractions of the energies carried by beams after emission of the initial state photons. The energy cut $\Theta(v_{max}-1+x_1x_2)$ limits the invariant mass[7] $\sqrt{s'}$, of the dressed (calorimetric) pair of $e^\pm$ because $1-s'/s = 1-x_1x_2 = v < v_{max}$. The angular trigger defined by the other $\Theta$'s may be asymmetric or symmetric, see Fig. 1. The Born cross section with pure $t$-channel photon exchange reads

$$\frac{d\sigma^{Born}}{d\xi^*} = \frac{2\pi\alpha^2}{sx_1x_2}\frac{1+(1-\xi^*)^2}{2\xi^{*2}}, \tag{5}$$

where $\vartheta^*$ in $\xi^* = (1-\cos\vartheta^*)/2$ is defined as the scattering angle in the LL hard scattering rest frame[8]. The $\xi_{1,2}$ are related to $\xi^*$ as follows

$$\xi_1 = \frac{x_2\xi^*}{x_2\xi^*+x_1(1-\xi^*)}, \qquad \xi_2 = \frac{x_1\xi^*}{x_1\xi^*+x_2(1-\xi^*)}. \tag{6}$$

All the above was kinematics and the QED perturbative LL calculation is located in the so called electron structure functions $D(x,\beta)$, which in QED can be, contrary to QCD, calculated with an arbitrary precision. In our calculations we shall mainly use the non-singlet (valence) structure functions calculated perturbatively in the O($\alpha^1$), O($\alpha^2$) and exact O($\alpha^\infty$) result, see refs. [14, 15, 16] for more discussion. The

[6] One should remember that it is not legitimate to divide subleading O($\alpha$) correction into pieces coming from initial and final state emission.

[7] Simple M.C. exercise shows that to a very high precision $x_i = E_i^{clust.}/E_{beam}$ where $E_i^{clust.}$ is the calorimetric energy of outgoing dressed $e^\pm$. Variables $x_i$ are therefore directly measurable!

[8] In low angle Bhabha $\vartheta^*$ is practically directly measurable because it can be trivially obtained from the laboratory (dressed) electron scattering angles $\vartheta_i$ through the approximate relation $\vartheta^* = \sqrt{\vartheta_1\vartheta_2}$

infinite order expression for non-singlet $D(x,\beta)$ reads

$$D(x,\beta) = \delta(1-x) + \beta P(x) + \frac{1}{2!}\beta^2\{P\otimes P\}(x) + \frac{1}{3!}\beta^3\{P\otimes P\otimes P\}(x) + \ldots \quad (7)$$

where $P(x) = \delta(1-x)(\ln\epsilon + 3/4) + \Theta(1-x-\epsilon)(1+x^2)/(1-x)$ and $\{P\otimes P\}(x) = \int_0^1 dx_1 \int_0^1 dx_2 \delta(x-x_1-x_2)P(x_1)P(x_2)$. In the LL $O(\alpha^2)$ calculation we use $D(x_1,\beta_t)D(x_2,\beta_t)$ truncated up to terms of $O(\beta_t^2)$. The corresponding formula reads

$$\begin{aligned}
D(x_1,\beta_t)D(x_2,\beta_t)|_{O(\frac{\alpha}{\pi})^2} &= \delta(1-x_1)\delta(1-x_2)(1+\Delta)\\
&+\delta(1-x_1)\,\theta(1-x_2-\epsilon)f_1(x_2) + \delta(1-x_2)\,\theta(1-x_1-\epsilon)f_1(x_1)\\
&+\theta(1-x_1-\epsilon)\,\theta(1-x_2-\epsilon)\,\beta_t^2\,\frac{1+x_1^2}{1-x_1}\,\frac{1+x_2^2}{1-x_2},
\end{aligned} \quad (8)$$

$$\Delta = \beta_t\,(\frac{3}{2}+2\ln\epsilon) + \frac{1}{4}\beta_t^2\,(\frac{3}{2}+2\ln\epsilon)^2 + \frac{1}{2}\beta_t^2\left(2\ln^2\epsilon + 3\ln\epsilon + \frac{9}{8} - \frac{\pi^2}{3}\right),$$

$$\begin{aligned}
f_1(x) &= \frac{1+x^2}{1-x}\left\{\beta_t + \frac{1}{2}\beta_t^2\left(\frac{3}{2}+2\ln\epsilon\right) + \frac{1}{2}\beta_t^2\left(2\ln(1-x) - \ln x + \frac{3}{2}\right)\right\}\\
&+\frac{1}{2}\beta_t^2\left(\frac{1}{2}(1+x)\ln x - 1 + x\right).
\end{aligned}$$

In fact, we have at our disposal an even better expression for the non-singlet structure function than the above $O(\alpha^2)$ result. In ref. [15] the following expression for the non-singlet structure function was given

$$\begin{aligned}
D_3^{YFS}(x,\beta_t) &= \frac{\exp\left(\beta_t(\frac{3}{4}-\gamma)\right)}{\Gamma(1+\beta_t)}\,\beta_t(1-x)^{\beta_t-1}\Big\{1 - \frac{1}{2}(1-x^2)\\
&+\beta_t\left[-\frac{1}{8}(1+3x^2)\ln(x) - \frac{1}{4}(1-x)^2\right] + \beta_t^2\,\Big[\frac{1}{8}(1-x)^2\\
&+\frac{1}{16}(3x^2-4x+1)\ln(x) + \frac{1}{96}(1+7x^2)\ln^2(x) + \frac{1}{8}(1-x^2)\,\mathrm{Li}_2(1-x)\Big]\Big\}
\end{aligned} \quad (9)$$

We treat it, here, at the level of our technical precision $\delta\sigma/\sigma = 0.02\%$, as identical with the infinite order exact solution. In fact, formula (9) represents the third order solution exponentiated according to the Yennie-Frautschi-Suura [6] prescription, see also [17, 15, 16, 18] [9]. Of course, the difference between the two above solutions for the non-singlet structure functions is dominated by the LL third order correction, an interesting quantity in itself.

In the following numerical calculations we integrate the phase space integral in eq. (5) with the M.C. method. The advantage of the M.C. integration is that such a calculation can be easily transformed into an event generator which provides

[9] It was cross-checked with two numerical programs, again different from those used in the present work, providing the $O(\alpha^\infty)$ solution for non-singlet electron structure function [15].

explicit four-momenta of the final particles. Such a generator may be then used to calculate the integrated cross section for arbitrarily complicated experimental cuts/acceptance. In fact, we provide to any interested user the M.C. event generator LUMLOG [19] for this purpose. The key point is the control of the technical precision. It can be achieved only by doing the numerical calculations with at least two methods as different and independent as possible[10]. Here, to integrate the phase space in eq. (5), we have developed (independently) two M.C. integration programs[11] with completely different M.C. algorithms. We have checked that their results agree to within $\delta\sigma/\sigma = 0.02\%$ for a wide range of angular and energy cuts. In addition, we have checked the M.C. programs against the analytical calculation by running them for structure functions $D(x,\beta)$ simplified to such an extent that the integral was feasible analytically. We conclude from these extensive comparisons that we control the technical/numerical precision of our LL results up to 0.02%.

In the following figures and table we shall examine, first, the dependence of QED correction on the type and strength of the angular and energy cut and, next, we shall look into the magnitude of the second (and third) order LL corrections. The LL version of the exponentiation will be also discussed.

In Fig. 2 we show the dependence of the LL results on the strength of energy cut $v_{max}$ for three types of the angular trigger. The correction $\delta_{LL}^{(\infty)} = \sigma_{LL}^{(\infty)}/\sigma_{Born} - 1$ is calculated using eq. (5) with the non-singlet structure function of eq. (9). (The shape of the curves would be similar for the analogous $O(\alpha)$ and $O(\alpha^2)$ corrections.) The striking result is that the curve for the asymmetric N-W cut lies above symmetric N-N and W-W curves. In fact, for strongly asymmetric angular cuts this curve may get shifted so high that for loose energy cut the QED correction (without vacuum polarization) may become positive, similarly as in the radiative tail of the narrow resonance! The other phenomenon is the occurrence of the plateau for high $v_{max}$. The starting point of plateau is $v_{max} \simeq 0.4$ for N-N and $\simeq 0.6$ for W-W cut. This reflects the simple kinematical effect [7]. Events with single hard photon (collinear to beam) are eliminated by the angular trigger cut because they have $\vartheta_1$ so different from $\vartheta_2$ that they cannot fit into an angular trigger any more[12]. Note that events with both beams emitting a hard photon are not affected by the angular cut and for this reason curves are rising slightly close to $v_{max} = 1$.

In fig. 3 we show our most important result, i.e., the difference between $O(\alpha^\infty)$ and $O(\alpha)$ LL corrections. Of course, this difference is dominated by the $O(\alpha^2)$ correction. It is interesting because the actual measurement of the luminosity in LEP is based on the $O(\alpha)$ calculation and the $O(\alpha^2)$ LL correction is the biggest QED bremsstrahlung correction not included in the luminosity QED calculation.

[10]In ref. [7] technical precision of the $O(\alpha)$ calculation was determined by comparing M.C. and semi-analytical calculations.

[11]These were unpublished programs MULTILOG and BHALOG which also contain sizeable testing parts. LUMLOG is more compact and within technical precision 0.02% is equivalent to them.

[12]Plateau starts approximately at $v_{max} = 1 - \vartheta_{min}/\vartheta_{\max}$.

| *trigger* | $\sigma_{Born}$ $[nb]$ | $\sigma^{(\infty)}_{LL}/\sigma_{Born}-1$ | $(\sigma^{(\infty)}_{LL}-\sigma^{(1)}_{LL})/\sigma_{Born}$ |
|---|---|---|---|
| Narrow-Wide | 28.8961 | $-0.02060 \pm 0.00012$ | $-0.00107 \pm 0.00010$ |
| Narrow-Narrow | 28.8961 | $-0.06330 \pm 0.00013$ | $0.00431 \pm 0.00010$ |
| Wide-Wide | 47.1439 | $-0.04474 \pm 0.00004$ | $0.00222 \pm 0.00007$ |

Table 2: *Numerical results at* $\sqrt{s} = 92GeV$ *for energy cut* $v_{max} = 0.5$. *The three angular triggers are those of Fig. 1. Big-log parameter* $\beta_t^{(A)}$ *is chosen. The cross section* $\sigma^{(\infty)}_{LL}$ *is defined in eq. (5) with the structure functions of eq. (9). The* $\sigma^{(1)}_{LL}$ *is defined in the same way except that for structure functions we use eq. (9) truncated to the first order in* $\beta_t$. *Results are from LUMLOG Monte Carlo [19] for* $5 \cdot 10^7$ *events. Statistical M.C. errors are shown.*

As seen in the figure the correction for symmetric cuts N-N and W-W is positive as expected from the "exponentiation type rule of thumb". *For asymmetric angular cut N-W the* $O(\alpha^2)$ *correction may be positive or negative depending on the strength of the angular and energy cuts*[13]. One has, therefore, to calculate this correction with the help of the M.C. calculation for the true experimental acceptance. It can be easily misleading to speak about the magnitude of the $O(\alpha^2)$ correction in luminosity measurements *in general*, not specifying the type and strengths of cuts or to generalize on its size from one numerical example. The shape of the curves in the figure we understand qualitatively. The rise at low $v_{max}$ is due to the standard negative infrared divergence in the (unexponentiated) $O(\alpha)$ cross section while the steady rise towards $v_{max} = 1$ is due to opening of two hard photon channel in the $O(\alpha^2)$ case. The figures analogous to Figs. 2 and 3 for the unexponentiated $O(\alpha^2)$ structure function looks very similar, therefore, we decided not to include them.

In Table 2 the results of Figs. 2 and 3 are also presented in a numerical form. They are shown for one value $v_{max} = 0.5$, corresponding roughly to the typical LEP energy cut, and for the same three angular triggers[14].

Fig. 4 illustrates an important observation on the exponentiation. It is known [20] that the $O(\alpha)$ exponentiated calculation is as precise as the $O(\alpha^2)$ unexponentiated one. Is the same true for the $t$-channel process like Bhabha or $ep$ scattering? To answer this question fully, we would need the $O(\alpha^2)$ exponentiated calculation which exists for the $s$-channel process [17] and is not yet available for Bhabha. We may give some hint, however, within the present LL initial state exercise. In Fig. 4 we plot the difference between $O(\alpha^2)$ unexponentiated result, obtained using the

[13]This is again very much reminiscent of the situation in the radiative tail of narrow resonance.

[14]The first order LL cross-section $\sigma^{(1)}_{LL}$ in Table 2 was cross-checked with several other methods, in particular with those which produced Table 1, after switching off the final state bremsstrahlung.

corresponding structure functions of eq. (9), and the $O(\alpha)$ exponentiated result, which is produced using the structure function of eq. (9) truncated to first order accordingly[15]. The difference is generally below 0.1%, even for high $v_{max}$ and asymmetric cut. In the same figure we also plot the "absolute error" of the $O(\alpha)$ exponentiated result, i.e, the difference between exact $O(\alpha^\infty) \simeq O(\alpha^3)_{exp}$ and the $O(\alpha)$ exponentiated results. It is below 0.05% with respect to Born. We conclude, therefore, that *it is quite likely that for low angle Bhabha the first order exponentiated QED calculation is as precise as the second order without exponentiation.* The first order exponentiated QED calculation (M.C.) is most probably sufficiently good for all practical purposes.

The final question to be discussed is: what is the practical relevance of the above LL calculations for the real experiment with real experimental cut, provided the analysis of the QED corrections was already done with help of the $O(\alpha)$ Monte Carlo? In this case we propose the following (temporary) method of reducing QED bremsstrahlung uncertainty down to 0.2% (together with pairs below 0.3%):

(a) Check the numerical precision of your $O(\alpha)$ M.C. program. Switch-off vacuum polarization and $s$-channel $Z$ exchange and verify if for simplified cut-offs M.C. cross-section agrees with tables/programs of ref. [7]. For $\vartheta < 10^o$ an agreement below 0.1% is required. You may typically find a bias due to the $k_0$ parameter [8, 9, 10].

(b) Switch on vacuum polarization and $Z$ exchange. As was noted in ref. [12] it is necessary to include the vacuum polarization also in the hard photon part (form factor due to Dyson summation)[16]. Calculate $O(\alpha)$ correction $\delta_1$ with the true experimental cuts using the detector simulation (DS) program.

(c) Calculate LL second (and higher) order LL corrections $\delta_2 = (\sigma^\infty - \sigma^{(1)}_{LL})/\sigma_{Born}$ by running the LL Monte Carlo LUMLOG [19] with the same true experimental cuts using *the same* DS program

(d) Combine $\delta_1 + \delta_2$ and use it to remove QED bremsstrahlung effect from the luminosity measurement.

What is the remaining QED uncertainty? The biggest QED bremsstrahlung correction to total cross section is now the next-to-leading $O(\alpha^2)$ correction, i.e. correction with generic coefficient $(\frac{\alpha}{\pi})^2 \ln(|t|/m_e^2)$ in front of it[17]. Let us try to estimate this term in various ways. The first estimate is simply $(2\frac{\alpha}{\pi})^2 2\ln(-t/m_e^2) \approx 0.05\%$. Another way is to assume that the second order subleading correction is roughly one fifth of the second order leading correction as it is the case in the first order, see

[15]More precisely we use $D_1^{YFS}(x,\beta_t) = \exp\left(\beta_t(3/4-\gamma)\right)/\Gamma(1+\beta_t)\ \beta_t(1-x)^{\beta_t-1}(1+x^2)/2$.

[16]By doing this one introduces the numerically important second and higher order LL correction $\simeq 0.002$ [8].

[17]Note that the LL third order correction with a generic coefficient of the same size is already included in $\delta_2$ and subtracted.

Table 1. This gives $\frac{1}{5}0.5\% = 0.1\%$. The third possibility is to look into variation of the value of $\delta_2 = (\sigma^{(\infty)} - \sigma^{(1)}_{LL})/\sigma_{Born}$ due to a change of definition of $\beta_t$. The change $\beta^{(A)}_t \rightarrow \beta^{(B)}_t$ induces the corresponding change of $\delta_2$ by $\approx 0.01\%$ for our N-N trigger and $\approx 0.02\%$ for W-W trigger. We conclude from all above estimates, taking safety factor two, that *the conservative upper limit for the second order subleading correction is 0.2%.* Of course, it would be better to calculate it rather than to estimate.

Another class of QED corrections is resulting from the real/virtual pair production. At the 0.1% precision level a systematic study of this correction for Bhabha scattering should be done. Here, we only make a simple extension of our LL exercise. In the LL approximation the introduction of light $e^{\pm}$ pairs is equivalent to the introduction of the running QED coupling constant into structure functions [14, 21, 20], i.e., $\beta^{(A)}_t \rightarrow -3\ln(1-(\alpha/3\pi)(\ln(|t|/m_e^2)-1))$. This is equivalent to adding real/virtual electron pairs to a non-singlet structure function in the LL approximation[18]. The $e^{\pm}$ pair effect obtained in this way varies from $-0.02\%$ to $-0.04\%$ depending on the type of the cut (N-W and N-N, $v_{max} = 0.5$). Since muon and quark pair LL contributions to the non-singlet structure function are a factor of three smaller and the singlet (sea) LL contribution is also typically smaller than the singlet one we, therefore, estimate the conservative upper limit on the light fermion pairs contribution to be 0.1%. *The total QED uncertainty we estimate to be below 0.3%, which includes 0.2% due to pure bremsstrahlung and 0.1% from light fermion pairs. Hadronic uncertainty of the vacuum polarization is not included here.*

Is it possible to lower this error in the future? The 0.1% QED bremsstrahlung uncertainty in low-angle Bhabha is in our opinion definitely achievable. This would require second order exponentiated calculation in a form of a single M.C. program (for example a second order extension of the BHLUMI program [5]) and more solid estimation/calculation of the second order subleading contribution and light fermion pair production. The additional hadronic uncertainty in vacuum polarization of 0.1% will remain with us for a long time [22], until we have new data on the low energy $e^+e^-$ annihilation cross section to feed into dispersion relations.

Let us finally return once again to the question of the final state bremsstrahlung corrections. As already noted, for the calorimetric measurement of the final state the Lee-Nauenberg-Kinoshita theorem [13] says that LL final state corrections cancel out. The remaining experimental details of the $e^{\pm}$ detection may only affect the subleading $O(\alpha)$ correction which in step (b) of our recipe is included properly. Any thinkable additional effects enter subleading second order corrections. Since the experimental detection of the final state $e^{\pm}$ may be sometimes very complicated we propose the additional cross-check on the above statements. If, for the true experimental acceptance, any LL final state effect would appear in the total QED

[18] For the cut $v_{max} \simeq 0.5$, as in luminosity measurement, one may check using existing $s$-channel calculations that the ratio of subleading to leading corrections is one to four [16]; we therefore believe that it is enough to use the LL approximation to calculate the pairs contribution in the small angle Bhabha scattering

correction then it would imitate abnormally high, say $\sim 2\%$ or bigger, subleading correction[19]. How does one check in practice if it is the case? One should proceed as follows: (i) calculate the O($\alpha$) correction with the O($\alpha$) M.C. without vacuum polarization and $Z$ exchange for true experimental cuts/acceptance and (ii) calculate the O($\alpha$) LL correction with *the same* detector simulation, using LUMLOG M.C. The difference of the two QED corrections is the subleading O($\alpha$) correction for these particular experimental cuts/acceptance! (Similarly as in our Table 1, but for real cuts.) The result should be of order 1% at most, otherwise the tested set of experimental cuts/acceptance requires additional study[20].

*To summarize,* the O($\alpha^2$) initial state LL correction is calculated with 0.02% technical precision for arbitrary cut-offs. It depends substantially on the type and strength of experimental cut. We provide a practical procedure, and the M.C. tools, to lower the QED precision for luminosity cross-section from 1% down to 0.3%. The way is open for testing second order M.C. programs for low angle Bhabha. We give a quantitative hint that for low angle Bhabha first order exponentiated calculation, as in BHLUMI Monte Carlo, is as good as the second order not exponentiated.

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Figure 2: *The dependence of the total LL correction* $\delta_{LL}^{(\infty)} = \sigma_{LL}^{(\infty)}/\sigma_{Born} - 1$ *on the energy cut* $v_{max}$ *for asymmetric angular trigger N-W and two symmetric trigger N-N and W-W, see Fig. 1. The cross section* $\delta_{LL}^{(\infty)}$ *is defined in eq. (5) with structure functions of eq. (9). Results are from LUMLOG/BHALOG Monte Carlo program for* $5 \cdot 10^7$ *events, statistical errors are below the size of dots.*

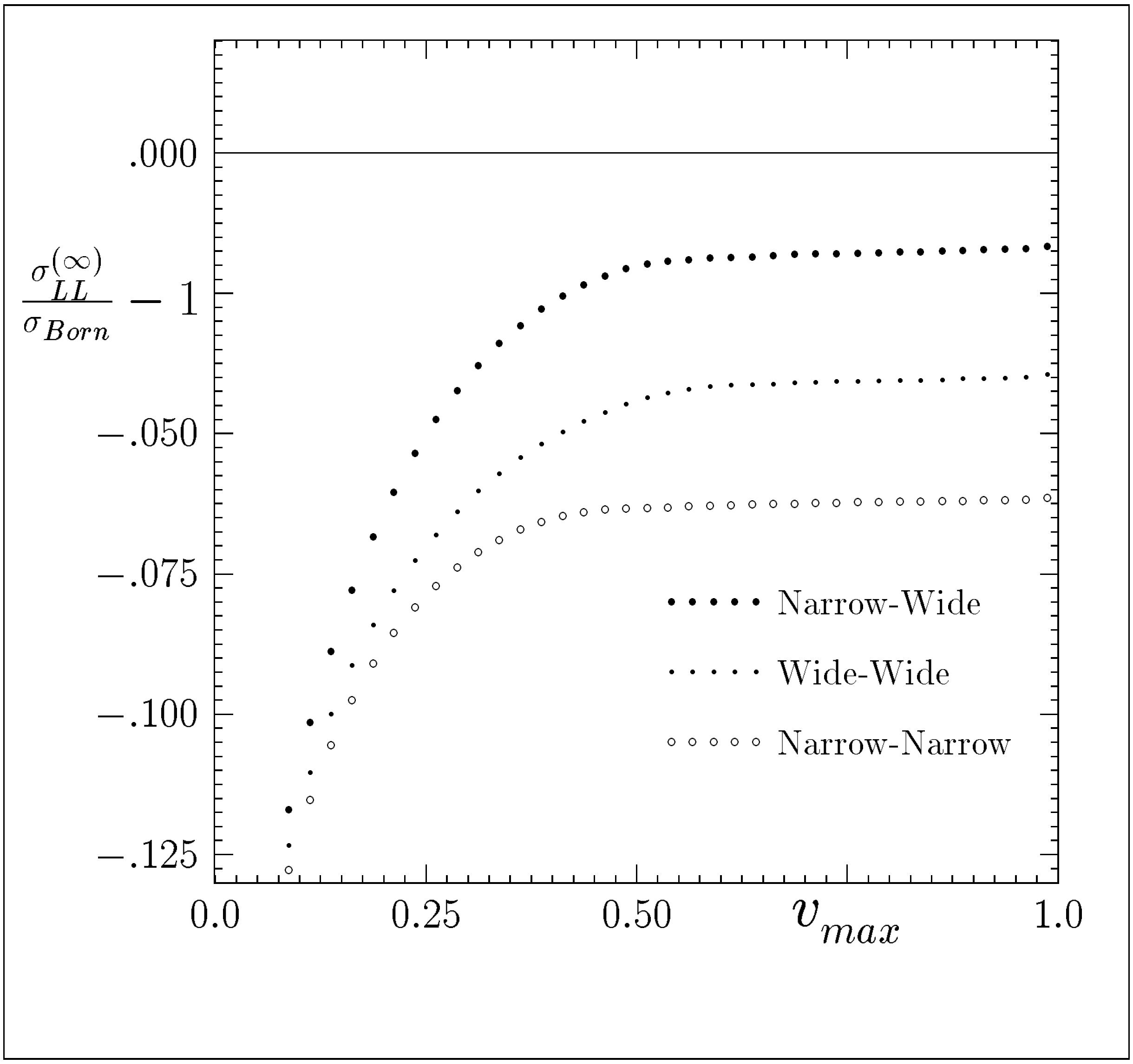

Figure 3: *The dependence of the difference between infinite and first order LL cross sections* $\delta_2 = (\sigma_{LL}^{(\infty)} - \sigma_{LL}^{(1)})/\sigma_{Born}$ *on the energy cut* $v_{max}$ *for asymmetric N-W angular trigger and two symmetric angular triggers N-N and W-W, see Fig. 1. Results are from LUMLOG/BHALOG Monte Carlo program for* $5 \cdot 10^7$ *events, statistical errors are below the size of dots.*

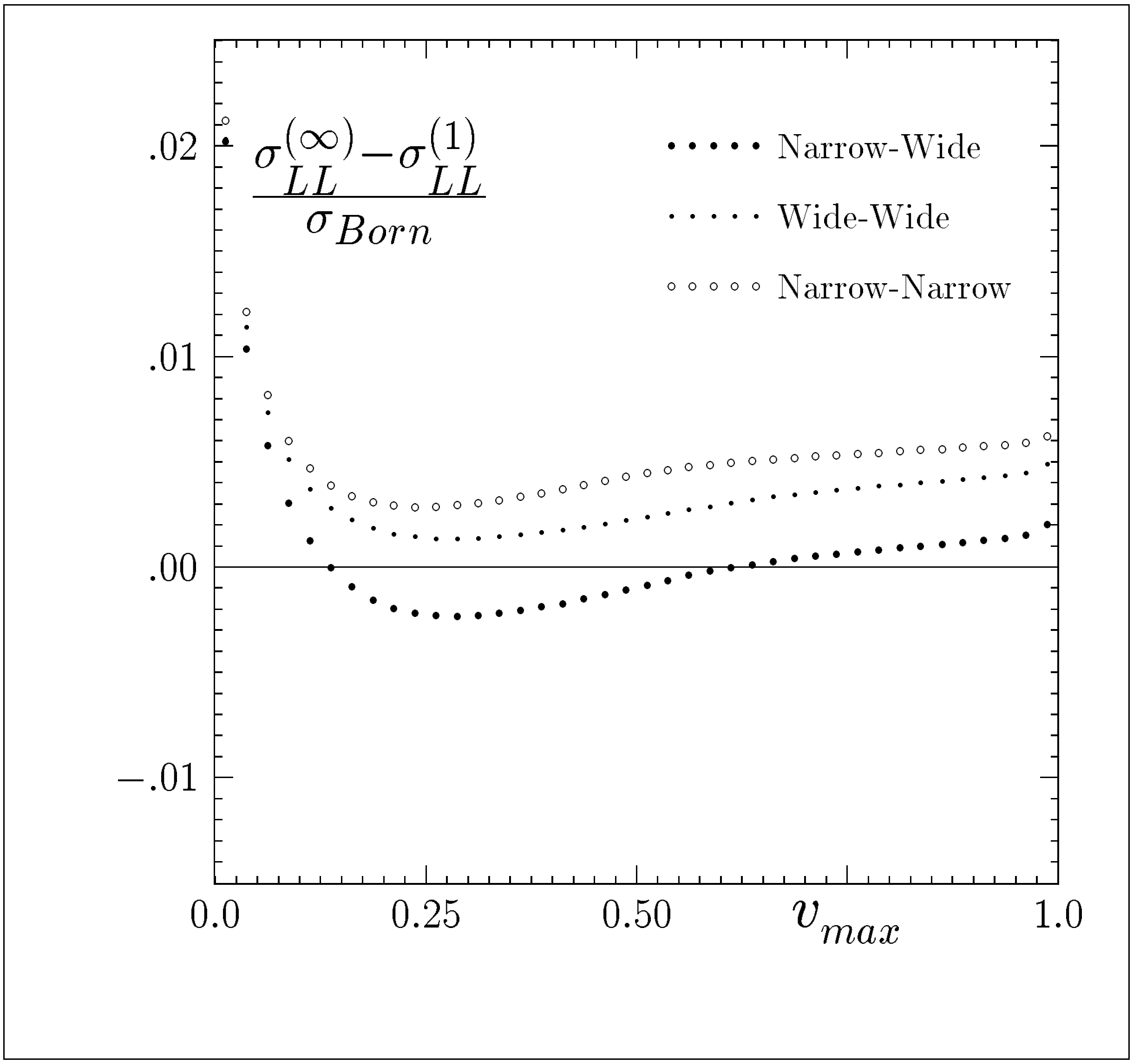

Figure 4: *The dependence of the difference between first order exponentiated and second order unexponentiated* $(\sigma^{(1)}_{LL,exp} - \sigma^{(2)}_{LL})/\sigma_{Born}$ *and the difference between exact and first order LL exponentiated* $(\sigma^{(\infty)}_{LL} - \sigma^{(1)}_{LL,exp})/\sigma_{Born}$ *results on the energy cut* $v_{max}$ *for symmetric (N-N) and asymmetric (N-W) angular triggers. Results are from LUMLOG/BHALOG Monte Carlo run with* $5 \cdot 10^7$ *events.*