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Yennie-Frautschi-Suura soft photons in Monte Carlo event generators

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ABSTRACT

It is shown how to simulate the production of multiple soft photons according to Yennie-Frautschi-Suura distributions in the Monte Carlo event generator. The photon multiplicity is approximately Poissonian with the average multiplicity depending on the detectability threshold in photon energy. This threshold may be set arbitrarily low. The presented methods may be applied in the Monte Carlo event generator for any QED process. An example of the computer code is included.

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1. Introduction

There are two basic reasons for the introduction of multiple soft photons in the Monte Carlo event generators for QED processes, that is for including them explicitly in the list of the randomly generated final state four-momenta. First, they are seen explicitly in the detector of a typical modern experiment, and it is therefore highly desirable to have them included in the M.C. generators which are commonly used to get predictions from theoretical models and to study the detector performance. Second, the inclusion of higher order QED corrections necessarily involves a higher number of emitted photons. Since it is known [1] how to sum up contributions from soft photons to any order it is therefore a resonable strategy to start this task with the inclusion of many soft photons. The next step in this direction would be an extension of the photon phase space to hard collinear configurations [2] and/or to the entire phase space.

The problem of summing up the contributions to the cross section from many, virtual and real, soft photons (so called exponentiation) has a long history and its most complete solution in the framework of the modern perturbative (covariant) QED may be found in ref. [1]. We also refere the reader to this paper for more references on the subject. Presently, the effects of QED bremsstrahlung are typically calculated using Monte Carlo (M.C.) numerical programs which feature single bremsstrahlung only (see ref. [3] for a review of these techniques). The aim of this paper is to combine the two approaches and to include the summation over an infinite number of soft photons in the M.C. simulation. As was already pointed out, this would be a relatively quick and easy method of introducing higher order QED effects in the M.C. simulation in an approximate way, and it would also provide a convenient basis for more complete calculations of higher order QED effects.

Let us still add one remark for readers more familiar with the M.C. techniques for QED bremsstrahlung calculations. In the typical, presently available, M.C. generator the infrared singularity point (in photon momentum space) is isolated from the rest of phase space using a standard cut-off $E_\gamma < E_{max}$. Inside a sphere defined by this cut-off the contributions to the cross section from virtual and real photons are combined analytically and in the M.C. simulation events corresponding to this part of the phase space (soft photon events) do not include, in the list of the generated four momenta, any bremsstrahlung photon at all, while the rest of events do include one energetic bremsstrahlung photon. When experimental precision and/or resolution increase one may be forced to lower the cut-off E_{max}. (In principle it is a dummy parameter which must not influence any calculated result). However, it cannot be lowered too much because the cross section for soft events $\sigma(E_\gamma < E_{max})$ will become negative. That would be a definite signal that a second bremsstrahlung should be taken into account. In the M.C. program of the type described in this paper the soft cross section never becomes negative and the cut-off on the combined real and virtual photons may be set arbitrarily low. The only effect will be an increase of the average multiplicity of soft photons. Setting the cut-off below a certain value (determined by experimental conditions) will change nothing in the calculation results because the additional soft photons will have energies far below detectability threshold resulting from resolution.

The lay-out of the paper is the following. In the next Section we recall some of the results of the basic reference [1] in a form suitable for the the description of our M.C. algorithms. In Section 3 we describe two examples of the M.C. algorithms for randomly generating energies of soft photons. The first is simpler and better suited for final state radiation and the second is more complicated and can be applied to initial state radiation with strongly varying cross section. Section 3 concludes this report with some comments on the practical importance of the results presented here. At the end of the paper we include the source code of the FORTRAN program implementing the two algorithms.

2. Yennie-Frautschi-Suura soft photons

In this section we shall recall some formulae from ref. [1] in a form suitable for the description of the M.C. algorithms presented in the next Section. Without loss of generality we shall concentrate on the initial state bremsstrahlung in e^+e^- annihilation. The infrared singularity will be regularized by an energy cut-off rather than by a photon mass.

Let us take, as a starting point, formula (2.13) of ref. [1], specialized to the initial state bremsstrahlung:

$$\begin{aligned}\sigma(K) &= \int_0^K d\epsilon \sum_{n=0}^{\infty} \frac{1}{n!}\frac{d\sigma_n}{d\epsilon} \\ &= \exp\left(2\alpha\frac{i}{(2\pi)^3}\int_{k_s}^{\infty}\frac{d^4k}{k}\left[\frac{2p_1-k}{k(2p_1-k)} - \frac{2p_2-k}{k(2p_2-k)}\right]^2\right) \\ &\sum_{n=0}^{\infty}\prod_{m=1}^{n}\int \frac{d^3k_m}{k_m}\theta(k_m-k_s)\theta(K-\sum_{m=1}^{n}k_m)\tilde{\rho}_n(k_1,...k_m)\end{aligned} \tag{2.1}$$

where p_1, p_2 are beam momenta, k_i are momenta of photons and

$$\tilde{\rho}_n(k_1,...k_m) = \sum_{perm(i_1,...i_n)}\sum_{r=0}^{n}\frac{1}{r!(n-r)!}\tilde{S}(k_{i_1}),...,\tilde{S}(k_{i_r})\tilde{\beta}_{n-r}(k_{i_{r+1}},...k_{i_n}). \tag{2.2}$$

Here the infrared singularity is regularised by a photon energy cut-off $k_s \to 0$, both for virtual and real photons. The factor

$$\tilde{S}(k) = -\frac{\alpha}{4\pi^2}\left(\frac{p_1}{kp_1} - \frac{p_2}{kp_2}\right)^2 \tag{2.3}$$

is associated with real soft photon emission. The upper limit on photon energy K is related to experimental resolution and may typicaly be of order $10^{-2}E$, where $E = \sqrt{s}/2$ is beam energy (centre of mass system is assumed tacitly). The distributions $\tilde{\beta}_n$ are infrared regular and for their formal definition we refere the reader to ref. [1]. The exponentiated virtual contribution is

$$\begin{aligned}B &= \frac{i}{(2\pi)^3}\int \frac{d^4k}{k}\theta(k_s-k)\left[\frac{2p_1-k}{k(2p_1-k)} - \frac{2p_2-k}{k(2p_2-k)}\right]^2 \\ &= -\frac{1}{2\pi}\left(2\frac{\alpha}{\pi}(\ln\frac{s}{m^2}-1)\ln\frac{E}{k_s} - \frac{1}{2}\frac{\alpha}{\pi}\ln\frac{s}{m^2}\right),\end{aligned} \tag{2.4}$$

where $p_i^2 = m^2$ and $s = 2p_1p_2 = 4E^2$. As an introductory exercise let us check cancelation of the infrared singularites, ie. the independence of integrated cross sections on the cut-off k_s. At the same time, we shall introduce notation and calculate some quantities useful in the following description of the M.C. algorithm. We start with the replacement

$$\theta(K - \sum_{m=1}^{n} k_m) \rightarrow \prod_{m=1}^{n} \theta(K - k_m), \tag{2.5}$$

assuming also from now on that K is a constant much smaller than $E = \sqrt{s}/2$. The cross section (1) then reads

$$\sigma(K) = \exp\left(-2\frac{\alpha}{\pi}(\ln\frac{s}{m^2} - 1)\ln\frac{E}{k_s} + \frac{\alpha}{2\pi}\ln\frac{s}{m^2}\right) \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{m=1}^{n}$$
$$\int\limits_{k_s<k_m<K} \frac{d^3k_m}{k_m} \sum_{perm(i_1,...i_n)} \sum_{r=0}^{n} \frac{1}{r!(n-r)!} \tilde{S}(k_{i_1}),...,\tilde{S}(k_{i_r})\tilde{\beta}_{n-r}(k_{i_{r+1}},...k_{i_n}). \tag{2.6}$$

The sum $\sum\limits_{n}(...)$ in above formula, taking advantage of its factorisability, may be rexpressed as follows

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{r=0}^{n} \frac{1}{r!(n-r)!}\left[\int\limits_{k_s<k<K} \frac{d^3k}{k}\tilde{S}(k)\right]^r \prod_{i=1}^{n-r} \int\limits_{k_s<k_i<K} \frac{d^3k_i}{k_i}\tilde{\beta}_{n-r}(k_1,...k_{n-r}) \\ = \exp\left(\int\limits_{k_s<k<K} \frac{d^3k}{k}\tilde{S}(k)\right) \sum_{n=0}^{\infty} \frac{1}{n!}\tilde{\beta}_n(k_1,...k_n)\end{aligned} \tag{2.7}$$

and since

$$\int\limits_{k_s<k<K} \frac{d^3k}{k}\tilde{S}(k) = -\frac{2\alpha}{\pi}\left(\ln\frac{s}{m^2} - 1\right)\ln\frac{k_s}{K}$$

we finally obtain

$$\sigma(K) = \exp\left(-\frac{2\alpha}{\pi}(\ln\frac{s}{m^2} - 1)\ln\frac{E}{K} + \frac{\alpha}{2\pi}\ln\frac{s}{m^2}\right) \sum_{n=0}^{\infty} \frac{1}{n!}\tilde{\beta}_n(k_1,...k_n). \tag{2.8}$$

As expected, the dependence on the infrared regulator parameter k_s has disappeared from the integrated cross section. In fact our result is the same as that of ref. [1] except that we miss the factor $F(\alpha A) = \exp(-\alpha AC)/\Gamma(1 + \alpha A)$. This, however, is entirely due to the enlargement of the photon phase space related to the replacement (5).

3. Two Monte Carlo algorithms

Let us now adress the following question: can the calculation presented in the former Section serve as a basis for the M.C. simulation of the multiple soft photon bremsstrahlung? The answer is positive and we are going to present two examples of M.C. algorithms.

We start with the simpler one, in which soft photons are limited to the range $k_s < k < K$. Furthermore, in the distribution (2.6) we retain in the sum $\sum_r$ only the first and the most singular term (with highest power of $\tilde{S}$) proportional to $\tilde{\beta}_0$. As usual, it is assumed that in a more complete M.C. the neglected terms will be included easily by the rejection technique. It is also assumed that the value of $K \ll E$ is small enough to neglect $\tilde{\beta}_0$ dependence on k_i. The above assumptions are justified in the case of the initial state bremsstrahlung with the Born cross section depending mildly on the centre of mass energy and, generally, for the final state bremsstrahlung. The requirement of the weak dependence of the cross section on s will be relaxed in the second M.C. algorithm example. The resulting, simplified cross section reads

$$\begin{aligned}\sigma(K) = \exp\left(\frac{2\alpha}{\pi}(\ln\frac{s}{m^2}-1)\ln\frac{k_s}{E}+\frac{\alpha}{2\pi}\ln\frac{s}{m^2}\right)\\ \sum_{n=0}^{\infty}\frac{1}{n!}\prod_{m=1}^{n}\int_{k_s<k_m<K}\frac{d^3k_m}{k_m}\tilde{S}(k_1)...\tilde{S}(k_n)\tilde{\beta}_0.\end{aligned} \tag{3.1}$$

The above formula is perfectly suited for M.C. simulation because it has a simple factorisable form of an "independent emission" type, where the momentum of every photon may be generated independently. The angular distribution for each photon is given by the factor $\tilde{S}(k)$ and the method of its generation may be taken from existing M.C. programs, for example from [4]. The integration over the photon direction

$$\int_{k_s<k_m<K}\frac{d^3k}{k}\tilde{S}(k) = \alpha A\int_{k_s}^{K}\frac{dk}{k}, \qquad \alpha A = \frac{2\alpha}{\pi}(\ln\frac{s}{m^2}-1) \tag{3.2}$$

leads to a simple $1/k$ distribution for the photon energy. The photon multiplicity is distributed according to Poisson distribution

$$\sigma(K) = \sum_{n=0}^{\infty}\sigma_n, \quad \sigma_n = \sigma_B\frac{1}{n!}\left(\alpha A\ln\frac{K}{k_s}\right)^n = \sigma_B e^{-\bar{n}}\frac{\bar{n}^n}{n!}, \tag{3.3}$$

where

$$\sigma_B = \tilde{\beta}_0 \exp\left(\frac{2\alpha}{\pi} (\ln \frac{s}{m^2} - 1) \ln \frac{k_s}{E} + \frac{\alpha}{2\pi} \ln \frac{s}{m^2} \right), \quad \bar{n} = \alpha A \ln \frac{K}{k_s}. \tag{3.4}$$

In the following we are going to describe in more detail the M.C. algorithm for generating sumultaneously the photon multiplicity n and photon energies $k_1, k_2, ..., k_n$ with $1/k_i$ distribution. It is an extention of the known algorithm for generating Poisson distribution and the k_i are obtained in this method as a sort of byproduct.

Before we go into more detail, let us first introduce a useful change of variables

$$\frac{1}{n!} \prod_{i=1}^{n} \int_{k_s}^{K} \frac{dk_i}{k_i} = \int\limits_{\ln k_s}^{\ln K} \prod_{i=1}^{n} dy_i \theta(\ln K - y_1)\theta(y_2 - y_1)...\theta(y_n - \ln k_s), \tag{3.5}$$

where $y_i = \ln k_i$ and the ordering is due to Bose statistics. Now, the prescription for the Poisson distribution is the following: Take a series of uniform random numbers $r_i \in (0,1)$ and calculate the sum $R_N = -\sum\limits_{i=1}^{N} \ln r_i$. R_N is calculated step by step, increasing N by one, until for the first time, for some N, $R_N > \bar{n}$. It may be prooven that $n = N - 1$ chosen in this way, is distributed precisely according to the Poisson distribution with average $\bar{n}$. Furthermore, although it may be not obvious, the (byproduct) variables

$$R_1, R_2, ..., R_n, \quad n = N - 1$$

are distributed in the interval (0,$\bar{n}$) according to a distribution

$$\prod_{i=1}^{n} dR_i \; \theta(\bar{n} - R_n)\theta(R_{n-1} - R_{n-2}) \; ... \; \theta(R_2 - R_1)\theta(R_1). \tag{3.6}$$

We only need to make a linear mapping

$$y_i = \ln K + (\ln k_s - \ln K)(R_i/\bar{n}), \tag{3.7}$$

in order to obtain variables y_i with proper distribution (5), or in terms of photon energies

$$k_i = K \left(\frac{k_s}{K} \right)^{\frac{R_i}{\bar{n}}}. \tag{3.8}$$

This completes the description of the first M.C. algorithm. Before we come to the second algorithm let us first comment on the various options for the soft photon phase space. In the above M.C. algorith we have used the very simple soft photon phase space

$$\prod_{i=1}^{n} \theta(K - k_i)\theta(k_i - k_s) \tag{3.9}$$

One might worry that for sufficiently large number n of soft photons their total energy may sum up to more than the centre of mass energy. This, however, may be totaly neglected if K is reasonably low because the tail of the photon multiplicity distribution is decreasing very rapidly. The other possible choice for the soft photon phase space (prefered in paper [1])

$$\theta(K - \sum_{i=1}^{n} k_i) \prod_{i=1}^{n} \theta(k_i - k_s) \tag{3.10}$$

is not prone to the above problem. In the M.C. it is rather easy to reduce the phase space from (9) to (10) by rejecting events which violate the $\sum_{i=1}^{n} k_i < K$ condition. In this way the characteristic factor $F(\alpha A)$ in the cross section of ref. [1] and in other references is restored. In fact, the ratio of the number of the accepted events to all raw events is precisely equal to $F(\alpha A)$ and may be obtained as a result of the M.C. integration*.

In ref. [1] the phase space (10) and even its more restricted version

$$\delta(\epsilon - \sum_{i=1}^{n} k_i) \prod_{i=1}^{n} \theta(K - k_i)\theta(k_i - k_s) \tag{3.11}$$

was clearly prefered over the simple one (9). The important reason is that in the case of the Born cross section $\tilde{\beta}_0$ depending strongly on centre of mass energy $\tilde{\beta}_0$ will depend effectively, in the case of the initial state bremsstrahlung, on $\epsilon = \sum_{i=1}^{n} k_i$ even for low energy soft photons. Although the first M.C. example is sophisticated enough for final state bremsstrahlung and for the initial state bremsstrahlung with almost constant cross section, it is in general highly desirable that in the M.C. program the integration over the ϵ variable is treated separately. In practice this means that in the M.C. algorithm (as in our second example) the variable ϵ is generated first, roughly according to the distribution $\tilde{\beta}_0(\epsilon)/\epsilon$ and next, for given ϵ and n, the soft

*The author has checked this equality in the M.C. calculation up to five significant digits.

photon momenta are chosen according to bremsstrahlung matrix element within the phase space domain (11).

Let us now proceed to a description of the M.C. algorithm featuring the above specification. We assume for the moment that the values of ϵ and n were chosen according to the distribution

$$\begin{aligned} \frac{d\sigma_n}{d\epsilon}(\epsilon) &= \exp\left(2\alpha A \ln\frac{k_s}{E} + \frac{\alpha}{2\pi}\ln\frac{s}{m^2}\right) I_n(\epsilon), \\ I_n(\epsilon) &= \frac{1}{n!}\prod_{i=1}^{n} \int\limits_{k_s<k_m<K} \frac{d^3k_i}{k_i}\delta(\epsilon - \sum_{i=1}^{n} k_i) S(k_1)...S(k_n)\tilde{\beta}_0(\epsilon) \end{aligned} \tag{3.12}$$

and we shall concentrate first on the method of generating $k_1, k_2, ..., k_n$ obeying the constraint $\epsilon = \sum\limits_{i=1}^{n} k_i$. The cases $n = 0, 1$ have to be treated separately, and the relevant distributions are

$$\begin{aligned} I_0(\epsilon) &= \delta(\epsilon)\tilde{\beta}_0(0), \\ I_1(\epsilon) &= \int\limits_{k_s<k<K} \frac{d^3k}{k}\delta(\epsilon - k)S(k)\tilde{\beta}_0(\epsilon) = \theta(K-\epsilon)\theta(\epsilon - k_s)\frac{\alpha A}{\epsilon}\tilde{\beta}_0(\epsilon). \end{aligned} \tag{3.13}$$

For $n \geq 2$ let us integrate over the photon angle, introduce $y_i = \ln k_i$ variables and absorb $n!$ by ordering photon energies

$$I_n(\epsilon) = (\alpha A)^n \tilde{\beta}_0(\epsilon) \int \prod_{i=1}^{n} dy_i \theta(\ln\epsilon - y_1)\theta(y_1 - y_2)....\theta(y_n - \ln k_s)\delta(\epsilon - \sum_{i=1}^{n} e^{y_i}). \tag{3.14}$$

It is known that for αA not exceding unity the fastest photon ($i = 1$) makes up for most of $\sum\limits_{i=1}^{n} e^{y_i}$ and we shall base our M.C. algorithm on that phenomenon. To this end, we have to transform the integral into a more convenient form. First, we introduce a new integration variable Y

$$\begin{aligned} I_n(\epsilon) = (\alpha A)^n \tilde{\beta}_0(\epsilon) \int dY \delta(Y - \ln\epsilon + y_1) \int \prod_{i=1}^{n} dy_i \\ \theta(\ln\epsilon - y_1)....\theta(y_n - \ln k_s)\delta(\epsilon - \sum_{i=1}^{n} e^{y_i}) \end{aligned} \tag{3.15}$$

and next we shift all y_i by Y, introducing $z_i = y_i + Y$,

$$\begin{aligned} I_n(\epsilon) =& (\alpha A)^n \tilde{\beta}_0(\epsilon) \int dY \int \prod_{i=1}^{n} dz_i \\ & \delta(\ln\epsilon - z_1)\theta(z_1 - z_2)....\theta(z_n - \ln k_s - Y)\delta(\epsilon - e^{-Y}\sum_{i=1}^{n} e^{z_i}). \end{aligned} \tag{3.16}$$

Now, the energy conserving δ is removed by the integrating over Y variable*

$$I_n(\epsilon) = \frac{1}{\epsilon}(\alpha A)^n \tilde{\beta}_0(\epsilon) \int dY \int \prod_{i=1}^{n} dz_i \\ \delta(\ln\epsilon - z_1)\theta(z_1 - z_2)....\theta(z_n - \ln k_s - Y_0(z_1...z_n)), \tag{3.17}$$

where

$$Y_0(z_1...z_n) = \ln(\epsilon^{-1}\sum_{i=1}^{n} e^{z_i}) \geq 0. \tag{3.18}$$

Formula (17) is well suited for the M.C. provided that the integration domain

$$\ln\epsilon = z_1 > z_1 > z_2... > z_n > \ln k_s + Y_0(z_1...z_n)$$

is enlarged a little bit by temporarily setting $Y_0 \equiv 0$. As usual, this will later be corrected by a rejection. As in the previous case the $z_2, z_3, ..., z_n$ are distributed uniformly in the domain

$$\ln\epsilon > z_2 > z_3... > z_n > \ln k_s$$

and they are mapped into $k_1, k_2, ..., k_n$ by the transformation

$$k_i = \epsilon e^{z_i} / \Big(\sum_{i=1}^{n} e^{z_i}\Big).$$

The values of $z_2, ..., z_n$ will come as a byproduct of randomly generating the photon multiplicity n. The constraint $z_n - \ln k_s - Y_0(z_1, ..., z_n) > 0$ is restored simply by checking whether the slowest photon obeys the relation $k_n > k_s$ and rejecting events which violate it.

So far we have merely assumed that ϵ and n were chosen according to (12) and we will now specify how this can actualy be done. It appears rather profitable, while generating ϵ and n, to keep again the (temporary) simplification $Y_0(z_1, z_2, ..., z_n) = 0$. The corresponding distributions are

$$\frac{d\sigma'_n}{d\epsilon}(\epsilon) = \exp\left(2\alpha A \ln\frac{k_s}{E} + \frac{\alpha}{2\pi}\ln\frac{s}{m^2}\right) I'_n(\epsilon),$$
$$I'_n(\epsilon) = \frac{(\alpha A)^n}{\epsilon}\tilde{\beta}_0(\epsilon) \int \prod_{i=2}^{n} dz_i \theta(\ln\epsilon - z_2)\theta(z_2 - z_3)...\theta(z_n - \ln k_s) \tag{3.19}$$
$$= \frac{\alpha A}{\epsilon}\tilde{\beta}_0(\epsilon)\frac{\big(\alpha A \ln(\epsilon/k_s)\big)^{n-1}}{(n-1)!},$$

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*Similar transformations were used in the M.C. program of ref. [5] for generating rapidities in multiple pion production.

where the prime denotes the modification $Y_0 \equiv 0$. The summation over n leads to a nice ϵ distribution

$$\frac{d\sigma'}{d\epsilon} = \exp\Big(\frac{\alpha}{2\pi}\ln\frac{s}{m^2}\Big)\left[\delta(\epsilon)\tilde{\beta}_0(0)\int\limits_0^{k_s}\rho(\epsilon')d\epsilon' + \theta(\epsilon - k_s)\rho(\epsilon)\tilde{\beta}_0(\epsilon)\right] \tag{3.20}$$

where

$$\rho(\epsilon)d\epsilon = \frac{\alpha A}{\epsilon}\Big(\frac{\epsilon}{E}\Big)^{\alpha A}d\epsilon; \tag{3.21}$$

see also ref. [6]. In the M.C. algorithm one has to generate ϵ in the $(0, K)$ range according to $\rho(\epsilon)\tilde{\beta}_0(\epsilon)$ distribution. One then sets $n = 0$, $\epsilon = 0$ for $\epsilon < k_s$, while for $\epsilon > k_s$ multiplicity $n > 0$ is generated according to Poisson distribution

$$P_{n-1} = e^{-\bar{n}}\,\frac{(\bar{n})^{n-1}}{(n-1)!}, \quad \bar{n} = \alpha A \ln\frac{\epsilon}{k_s}. \tag{3.22}$$

As already mentioned, $n-1$ variables used to generate P_{n-1} may be used to chose photon energies $k_1, ..., k_n$ such that they obey the constraint $\sum\limits_i k_i = \epsilon$. If the above method is used then one only has to remember that the rejection which is necessary for correcting for the simplification $Y_0 \equiv 0$ (reject events with $k_n < k_s$) should be done for the entire event, that is in the case of rejection one has to restart the procedure with new ϵ and n.

4. Conclusions

In this report we present, to our knowledge, the first consistent attempt of incorporating in the M.C. program the multiple soft photon production following the well established phenomenon of exponentiation of the soft photon cross sections. In both of the presented M.C. algorithms soft photons are produced in the range (k_s, K) where k_s can be set arbitrarily small, below detectability threshold resulting from a given experimental resolution and K should be small in comparison with $E = \sqrt{s}/2$. The multiplicity distribution of soft photons is approximately Poissonian but slightly deformed due to an energy conservation requirement. The average photon multiplicity is proportional to $\log(K/k_s)$ and, therefore, for very low k_s may be arbitrarily high. In this case, however, most of the soft photons will carry such litle energy, that in any case they will be unable to influence any observable quantity.

The practical importance of the presented methods of including exponentiation in the Monte Carlo is the following. First, they may be used to supplement easily the existing M.C. programs which feature single bremsstrahlung with a possibility of generating multiple soft photons in a simplified way, for example, by generating only single bremsstrahlung above K and multiple bremsstrahlung below. This may prove sufficient in many situations where only approximate estimates of multibremsstrahlung effects are required (background studies). If, however, a more complete treatment of the second bremsstrahlung is required then the above M.C. may be gradualy improved by adding more subtle effects by the rejection method (main singularities are already included in M.C.). Again, when the inclusion of the complete double bremsstrahlung (two loop) corrections is accomplished, the effect of any further soft bremsstrahlung can be kept under control and easily estimated.

At the end of this report we include the source code of two FORTRAN subprograms which may be used to randomly generate multiplicity and energies of the soft photons in the predefined range according to two M.C. algorithms discussed in Section 3. Four-momenta of photons are not explicitly generated. This may be done using subrograms from any M.C. program for QED bremsstrahlung. In the second M.C. algorithm (photons with restricted sum of energies) $\tilde{\beta}_0(\epsilon)$ is assumed to be constant in order to be able to compare the results, for example photon spectra, from both methods. It was found that they agree numerically, as expected.

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