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Monte Carlo simulation of the process $e^+e^- \to \tau^+\tau^-, \tau^\pm \to X^\pm$ including radiative $O(\alpha^3)$ QED corrections, mass and spin effects

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PROGRAM SUMMARY

Title of the program: KORAL-B

Computer: IBM; *Installation:* IBM 4341

Operating system: VM/SP R2

Programing language used: ANSI FORTRAN

High speed storage required: 45000 words

No. of bits in a word: 32

Peripherals used: Line printer

No. of cards in combined program and test deck: 1339

Card punching code: EBC DIC

Keywords: Radiative corrections, heavy lepton τ, Monte Carlo simulation, Quantum Electrodynamics, spin polarization, electroweak effects.

Nature of physical problem: In the process $e^+e^- \to \tau^+\tau^-(\gamma), \tau^\pm \to X^\pm$ the effects due photon radiation, spin polarization and finite mass of τ lepton are of similar magnitude [1, 2]. They should be taken into account in the analysis of the experimental data in order to subtract the trivial phenomena from the interesting ones like electroweak effects or lepton universality.

Method of solution: The Monte Carlo simulation of the combined τ production and decay process is used to analyse the spin effects and effects of radiative corrections, including hard bremsstrahlung, simultaneously. Any experimental cut and apparatus efficiency may be introduced easily by rejecting some part of the generated events. A new method of calculating numericaly QED distributions is developed.

Restrictions on the complexity of the problem: Incoming $e^\pm$ and outgoing $\tau^\pm$ may have arbitrary (transverse and longitudinal) polarizations. The total center of mass energy is restricted to the range from the τ production threshold to about 45 GeV. At higher energies the contribution from the Z_0 exchange should be treated in a more complete way. Only one prong decays of τ are included. The program can simulate the production of heavy quarks b and c (without fragmentation) in the same energy range.

Typical running time: Efficiency for unpolarized $e^\pm$ is 120 events per CPU sec. For polarized $e^\pm$ the efficiency depends on the longitudinal components of the $e^\pm$ polarizations. For transverse polarized $e^\pm$ it is 13 events per CPU sec.

references:

[1] S. Jadach and Z. Was, "QED O(α^3) radiative corrections ...", Jagellonian Univesity preprint TPJU-14/83, 1983, to be published in Acta Phys. Polonica.

[2] F.A. Berends and R. Kleiss, Nucl. Phys. **B177** (1981) 237.

LONG WRITE-UP

1. Introduction

The experiments in the electron-positron storage rings gradually acumulate higher statistics data on the production of the heavy lepton τ. The precise measurements of its properties is important in verifying the conjecture of the universality among the fermion families. The evidence of any deviation from the universality would give an important hint for solving the outstanding problem of the family replication.

The analysis of the experimental data on the τ pair production in the electron positron annihilation process requires taking into account the quantum electrodynamical radiative corrections which are known to give the important contributions to any lepton and quark production at PETRA/PEP energy range [1, 2] and [3]. It was found that the Monte Carlo (M.C.) simulation is an efficient and convenient method of calculating the effects of radiative corrections for $e^+e^- \to \mu^+\mu^-$ process [3, 4]. The main advantage of the M.C. method is that any kinematical cut introduced in a particular experiment may be easily introduced in the M.C. simulation, in the same way as in the real experiment, i.e. by rejecting some class of the M.C. events. Furthermore, for the combined τ production and decay process the final state phase space often includes many particles. In that case the M.C. simulation appears to be the best method to integrate (exactly) over the momenta of the final state particles.

The M.C. programs of refs. [3, 4, 5] calculate the cross sections and distributions for the process $\mu^+\mu^- \to \mu^+\mu^-(\gamma)$ with precission 1% or better using the approximation of the small muon mass. While analysing high statistics τ pair production data these programs cannot be used to attain the same accuracy level due to finite τ mass and spin effects in the τ decays. These effects are of order of a few percents [1] and there is practically no way to separate them from the radiative corrections.

The program presented here includes the conventional QED, spin and mass effects which may produce a variation of the distributions and cross sections by 1% or more. It is aimed to serve in the analysis of the high statistics τ pair production data as a tool helpful in eliminating the influence of the trivial QED, spin and mass effects on the measurements of the interesting physical quantities like coupling constants and branching ratios of τ , mass of ν_τ etc. More precisely, the present version of the program simulates the process $e^+e^- \rightarrow \tau^+\tau^-(\gamma), \tau^\pm \rightarrow \nu_\tau\mu^\pm(e^\pm,\pi^\pm,\rho^\pm)$ taking into account the following effects : 1. Order α^3 QED radiative corrections including hard bremsstrahlung from initial and final state fermions, 2. Arbitrary spin polarization (longitudinal, transverse and any other) for initial e^+ and e^-, 3. The decay of each τ in its rest frame taking into account its spin density matrix, spin correlations among two τ's etc., 4. The exchange of the electroweak boson Z_0 in a form suitable for the PETRA/PEP energy range and polarization effects which are due to Z_0 exchange.

The 1% accuracy level, the same as in the corresponding M.C. programs for the muon pair production process [3, 5], is assured for the energy range $2m_\tau \leq \sqrt{s} \leq 45$ GeV. To improve further on the precission one should include in future $O(\alpha^4)$ QED and/or the radiative corrections of the electroweak theory.

Subprograms simulating typical one-prong decays of τ are included in the test deck ($\tau \rightarrow \pi\nu, \rho\nu, e\nu\bar{\nu}, \mu\nu\bar{\nu}$) but, since the program has a transparent modular structure, the user may replace them easily with his own more sophisticated routines simulating these processes or with subprograms simulating other decays like multiprong decays, cascade decays, rare decays etc..

This program may be also used to simulate the production of the muon or any quark in the PETRA/PEP energy range. In particular it is recommended to simulate the production of the heavy quarks c and b for which the mass effects are as important as for lepton τ. For more details see Section 4.

We now outline briefly the simulation algorithm. First an event is generated assuming for the moment, that the incoming $e^\pm$ are not polarized and also taking zero spin polarizations in $\tau^\pm$ decays. In a second step all spin effects due to spin polarizations of incoming electron and positron and due to a dependence of the τ decay on its polarization are introduced by rejecting some part of the generated events. This method was outlined in ref. [1].

In the first, spinless, part of the algorithm we use the results of ref. [2] where the QED $O(\alpha^3)$ radiative corrections to $e^+e^- \rightarrow \tau^+\tau^-$ process were recalculated keeping a finite mass of the τ and using the $m_e << \sqrt{s}$ limit. This part of the algorithm is to some extent similar to that presented in ref. [5]. First, the photon momentum is generated and then it is decided whether photon was emitted from the initial or final state. Next, the angular variables are generated. The generation of the angular variables is changed with respect to ref. [5]. We employ a method which suits better the case of massive final fermions. Mass terms spoil a useful factorisation of the hard bremsstrahlung cross section into fragmentation function of the fermion into photon plus fermion and into the lowest order angular distribution in certain reference frames which was extensively used in ref. [5].

The important novel feature of the presented program with respect to programs of the type presented in ref. [5] is the fact that in the introduction of the spin effects in the second part of the algorithm the spin amplitudes of the radiatively corrected τ production process are calculated *numericaly* as a complex numbers and later the calculation of the cross section for the combined τ production and decay process involving squaring spin amplidudes and contracting them with spin density matrices of external fermions is also done numericaly in complex arithmetics. This powerful techniques proposed in ref. [1] and similar to that described in ref. [6] takes the advantage of the fact that the analytical expressions for spin amplitudes are far much simpler and easier to control than these for the differential cross sections. Squaring and spin-summing introduces enormous number of the interference terms. The above method, at first sight seems to contradict the requirement of the modular structure of the algorithm (program) assuring a good separation of the τ production and decay parts. Fortunately the acceptable solution can be found [1] relying on the consequent use of the spin density matrix formalism. As a result the production and decay parts of the program are clearly separated and it is rather easy to change the type of the τ decay without any change in the first, spinless production part, of the M.C. algorithm. All this techniques may find in future many applications in the M.C. simulation of the spin effects in other process: it generalises easily to higher spins and to cascade decays.

In the program described in this write-up the exchange of the electroweak boson Z_0 is included only in the lowest order and there are no radiative corrections to its exchange. This approximation is sufficient for PETRA/PEP energy range $\sqrt{s} \leq 45$ GeV. A future extension of the program should include Z_0 exchange in full resonance form and with QED radiative corrections to its exchange as in ref. [5] and the radiative corrections of the electroweak theory.

The layout of the write-up is as follows. In Section 2 we describe the event generation in the spinless case. Section 3 deals with spin effects. In Section 4 we describe the structure of the program, subprograms, common blocks and important variables. In Section 5 a test run and its output is presented.

2. The Monte Carlo algorithm without spin

In this Section we describe the methods used to generate M.C. events for the process

$$e^+(p_1) + e^-(p_2) \to \tau^+(q_1) + \tau^-(q_2) + \gamma(k), \tag{2.1}$$

in the spinless case.

The algorithm is summarized in fig. 1 in a pictorial way, with rectangular boxes denoting generation of the variables and romboid decision boxes.

In the first part of the algorithm (fig. 1, points 1,2,3) the photon momentum E_γ is generated according to the distribution

$$\frac{1}{\sigma_p}\frac{d\sigma}{dk} = \delta(k)\rho_s + \theta(k-k_0)\rho(k), \tag{2.2}$$

where $k = E_\gamma/E$ is the center of mass system photon momentum in units of the beam energy E, and the functions $\rho_s(k)$ and $\rho(k)$ will be defined later in this Section. The pointlike cross section $\sigma_p = (qq')^2\pi\alpha^2/3E^2$, where q and q' are charges of e^+ and of τ^+ (both equal to one), is often used as a normalisation constant in the program and in this write-up. The cut-off k_0 on the photon energy is used to isolate the infrared singularity. Events with $k \le k_0$ we shall call soft events. We assign them zero photon four momentum. For events with $k > k_0$, referred to as hard events, the decision is made (fig. 1, point 4) whether hard photon is emitted from the initial or final state fermions. The algorithm splits next into three main branches corresponding to three types of events i.e. soft events and two types of hard events, see fig. 1, points 5-15. The angular variables used to parametrize the phase space are generated in different way for every type of events. In the later stage of the generation (fig. 1, p. 16 and 17) the interference between the initial and final state radiation is added by rejection technique.

We now come to more detailed description of the algorithm. It starts with the generation of the center of mass photon momentum k. Its distribution (2.2) was given in ref. [2] with

$$\begin{aligned}
\rho(k) &= \rho_{ini}(k) + \rho_{fin}(k), \\
\rho_{ini}(k) &= \beta_{ini}\frac{1+(1-k)^2}{4k(1-k)}v(k)(3-v^2(k)), \\
\rho_{fin}(k) &= \frac{\alpha}{\pi}(q')^2\frac{1}{k}\Big[(1+(1-k)^2 - M^2k - \frac{1}{2}M^4)\ln\left(\frac{1+v(k)}{1-v(k)}\right) \\
&\quad - v(k)(k^2 + (2+M^2)(1-k))\Big],
\end{aligned} \tag{2.3}$$

where

$$M = m_\tau/E, \quad m = m_e/E, \quad v(k) = (1 - M^2/(1-k))^{1/2},$$
$$\beta_{ini} = 2\frac{\alpha}{\pi}q^2(\ln\frac{4}{m^2} - 1),$$

m_e and m_τ are electron and τ masses and α is the QED coupling constant. The interference of the initial and final state bremsstrahlung gives zero contribution to $d\sigma/dk$.

Photon momentum is generated as other variables in this algorithm by a weighting rejecting technique (WRT). It consists of finding some distribution $\tilde{\rho}(k)$ such that it is 1. simple, 2. as close as possible to $\rho(k)$, especially in the regions where $\rho(k)$ has its maxima, 3. obeying inequality $\rho(k) \leq \tilde{\rho}(k)$. Variable k is generated according to the (simple) distribution $\tilde{\rho}(k)$, refered to as dominant density (DD) distribution for $\rho(k)$. Next for every generated k the weight $w(k) = \rho(k)/\tilde{\rho}(k)$ is compared with the uniform random number $r \in (0,1)$, see fig. 1 point 3. The event is rejected if $r > w$, otherwise it is accepted. Accepted events are distributed according to $\rho(k)$. The average weight ¡w¿, calculated from both rejected and accepted events may be used to calculate the integral over $\rho(k)$ through the useful relation

$$\int \rho(k)dk = < w > \int \tilde{\rho}(k)dk. \tag{2.4}$$

This weighting rejecting (WR) technique generalises easily to multidimensional distributions and is also used in our algorithm to generate the angular distributions.

We find for $\rho(k), \quad k > k_0$, the following DD distribution

$$\tilde{\rho}(k) = \beta_{ini}\frac{1+(1-k)^2}{2k(1-k)} + \frac{\alpha}{\pi}(q')^2\frac{3}{2}\frac{1+(1-k)^2-2M^2}{k}\ln\left(\frac{1+v(0)}{1-v(0)}\right). \tag{2.5}$$

The value of k is generated using the same self learning procedure as in ref. [5] according to the distribution

$$\frac{1}{\sigma_p}\frac{d\sigma}{dk} = \delta(k)\rho_s + \theta(k-k_0)\tilde{\rho}(k). \tag{2.6}$$

The rejection is done only for hard events using the weight

$$w_k = \rho(k)/\tilde{\rho}(k) \tag{2.7}$$

and the integrated cross section in the spinless case is obtained using relation (2.4)

$$\sigma = \sigma_p\Big(\rho_s + < w_k > \int_{k_0}^{1-M^2} dk\tilde{\rho}(k)\Big), \tag{2.8}$$

where the average weight is calculated using all hard events, including rejected events.

At this point, after the generation of the photon momentum k, we decide to which of the three classes the event belongs. If $k < k_0$ then it is a soft event. For $k > k_0$ we compare the uniform random $r \in (0,1)$ with the ratio

$$R_{ini} = \rho_{ini}/(\rho_{ini} + \rho_{fin}) \tag{2.9}$$

and if $r \leq R_{ini}$ $(r > R_{ini})$ we assume that the hard photon was emitted from initial (final) state, see fig. 1 point 4.

For soft events, $k < k_0$, (fig.1, points 5-7 and 16) the two body phase space is parametrised using the standard polar and azimuthal angles ϕ and θ of the τ^+ momentum in the c.m. system and taking z-axis along e^+ momentum. We split the differential soft cross section [1, 2]

$$\begin{aligned}
&d\sigma_{soft} = \frac{3\sigma_p}{8\pi}A_{soft}(c)d\tau_2,\\
&d\tau_2 = d^3q_1 d^3q_2 (E_1E_2)^{-1}\delta^4(p_1+p_2-q_1-q_2) = \frac{1}{2}\beta' dc d\phi,\\
&c = \cos\theta, \quad \beta' = (1-M^2)^{1/2},
\end{aligned} \tag{2.10}$$

into c-even and c-odd component and the function

$$A_{soft}(c) = A_{even}(c) + A_{odd}(c) \tag{2.11}$$

will be defined below.

The c-even part

$$A_{even}(c) = (1 + (\beta' c)^2 + M^2)(1 + \delta^{SX} - 2\mathrm{Re}\Pi + \delta^Z) + 4\mathrm{Re}\tilde{F}_2 \tag{2.12}$$

contains the contributions from the lowest order, from the vacuum polarization function $\Pi(s)$, the combined contribution from the initial and final state vertex functions F_1, $\tilde{F}_1$ and from c-even part of the soft photon bremsstrahlung

$$\begin{aligned}\delta^{SX} &= \frac{2\alpha}{\pi} q^2 (\ln\frac{4}{m^2} - 1)\ln k_0 + \frac{\alpha}{\pi} q^2 (\frac{3}{2}\ln\frac{4}{m^2} - 2 + \frac{\pi^2}{3}) \\ &+ \frac{2\alpha}{\pi}(q')^2 \Big(\frac{1+\beta'^2}{2\beta'}\ln\frac{(1+\beta')^2}{M^2} - 1\Big)\ln\frac{2k_0}{M} + \frac{\alpha}{\pi}(q')^2\Big(\frac{3+2\beta'^2}{2\beta'}\ln\frac{(1+\beta')^2}{M^2} \\ &\qquad - 2 - \frac{1+\beta'^2}{2\beta'}\Big[\ln^2\frac{(1+\beta')^2}{M^2} + 4\mathrm{Li}_2\Big(\frac{2\beta'}{1+\beta'}\Big) - \pi^2\Big]\Big),\end{aligned} \tag{2.13}$$

from the final state magnetic vertex function

$$\tilde{F}_2 = -\frac{\alpha}{\pi}(q')^2 \frac{M^2}{4\beta'}\Big(\ln\frac{1+\beta'}{1-\beta'} + i\pi\Big), \tag{2.14}$$

and from the Z_0 exchange

$$\delta^Z = 2\chi v\tilde{v}/(qq'). \tag{2.15}$$

The electroweak boson Z_0 with mass M_Z couples to electron with the vertex $-ie(a+v\gamma_5)\gamma^\mu$ and to the lepton τ with $-ie(\tilde{a}+\tilde{v}\gamma_5)\gamma^\mu$. The function

$$\chi = (p_1+p_2)^2/(M_Z^2 + (p_1+p_2)^2) \tag{2.16}$$

is the ratio of the photon and Z_0 propagators. Finally, Li_2 denotes dilogarithm function [7].

By integrating eq. (2.9) over the phase space we obtain the soft cross section $\sigma_s = \sigma(k < k_0)$ and the parameter ρ_s in eq. (2.6)

$$\sigma_{soft} = \sigma_p \rho_s = \sigma_p \beta' \Big[\, (1 + \frac{1}{2}M^2)(1 + \delta^{SX} - 2\mathrm{Re}\Pi + \delta^Z) + 3\mathrm{Re}\tilde{F}_2\Big]. \tag{2.17}$$

To produce an event according to the distribution $A_{soft}(c)$ we choose first $\phi \in (0, 2\pi), c \in (-1, 1)$ with constant probability. Then in the first step we reproduce $A_{even}(c)$ by using WR technique with the weight

$$w_s = A_{even}(c)/A_{even}(1), \tag{2.18}$$

see fig. 1, p. 6,7. In the next step (fig. 1, point 16) another WR procedure is employed to add A_{odd} term to the generated distribution. In this second step generated events are rejected using the weight

$$w^s_{int} = \frac{1}{2}(A_{even} + A_{odd})/A_{even}. \tag{2.19}$$

The function

$$A_{odd}(c) = qq'\frac{\alpha}{\pi}\Big[(1 + (\beta' c)^2 + M^2)\delta_{BREM}(c) + \delta_{BOX}(c) + \delta_{Z_0}(c)\Big] \tag{2.20}$$

contains the contributions from from c-odd part of the soft bremsstrahlung and from two box diagrams [1, 2]

$$\begin{aligned}
\delta_{BREM}(c) &= 4\ln\frac{1-\beta' c}{1+\beta' c}\ln k_0 + D(c) - D(-c),\\
\delta_{BOX}(c) &= Re(Z_1 + cZ_2 + M^2 s Z_3),
\end{aligned} \tag{2.21}$$

where

$$\begin{aligned}
D(c) = {} & 2Re\Big[\mathrm{Li}_2\left(\frac{-x}{1-y}\right) - \mathrm{Li}_2\left(\frac{1-x}{1-y}\right) - \mathrm{Li}_2\left(\frac{1+x}{y}\right) + \mathrm{Li}_2\left(\frac{x}{y}\right)\Big]\\
& + \ln^2\left(\frac{y}{1-y}\right) - Re\mathrm{Li}_2(x^2) + \frac{1}{2}\ln^2(x^2) - \ln(x^2)\ln(1-x^2),\\
x = {} & (1-\beta' c)/(2 - 2\beta' c - M^2)^{\frac{1}{2}},\\
y = {} & \frac{1}{2}((2 - 2\beta' c - M^2)^{\frac{1}{2}} + M^2/(1+\beta')),\\
c = {} & \cos\theta, \quad s = \sin\theta
\end{aligned}$$

and the contribution from Z_0 exchange

$$\delta_{Z_0} = 4c\chi a\tilde{a}/(qq'), \tag{2.22}$$

where the three functions Z_i will be defined in Section 3 together with box spin amplitudes.

In the hard part of the algorithm (fig. 1, p. 8-15) three body (five dimensional) phase spase

$$d\tau_3 = \frac{d^3q_1 d^3q_2 d^3k}{E_1 E_2 E_\gamma}\delta^4(p_1 + p_2 - q_1 - q_2 - k) \tag{2.23}$$

has to be used. The hard bremsstrahlung differential cross section

$$d\sigma = \frac{\alpha^3 (qq')^2}{16\pi^2 E^2}(A_{ini} + A_{fin} + A_{int}) d\tau_3 \tag{2.24}$$

is taken from refs. [1, 2]. The three components A_{ini}, A_{fin} and A_{int} represent hard photon radiation from the initial state, final state and their interference and they read

$$\begin{aligned}
A_{ini} &= \frac{q^2}{\tilde{s}_1 x_1 x_2}\Big[D_1(\tilde{t},\tilde{u})\left(1 - \frac{m^2 x_1}{\tilde{s}_1 x_2}\right) + D_1(\tilde{t}_1,\tilde{u}_1)\left(1 - \frac{m^2 x_2}{\tilde{s}_1 x_1}\right)\Big],\\
A_{fin} &= \frac{q'^2}{\tilde{s}_1 y_1 y_2}\Big[D_2(\tilde{t},\tilde{u}_1)\left(1 - \frac{M^2(y_1+y_2)}{\tilde{s} y_2}\right) + D_2(\tilde{t}_1,\tilde{u})\left(1 - \frac{M^2(y_1+y_2)}{\tilde{s} y_1}\right)\\
&\quad + \frac{M^2(x_1^2+x_2^2)}{\tilde{s}} - 4M^2(\tilde{s}-\tilde{s}_1)\Big],\\
A_{int} &= \frac{qq'}{\tilde{s}\tilde{s}_1 x_1 x_2 y_1 y_2}\Big[M^2 x_1 x_2[(\tilde{s}-\tilde{s}_1)(\tilde{t}+\tilde{t}_1-\tilde{u}-\tilde{u}_1) - (x_1-x_2)(y_1-y_2)]\\
&\quad (\tilde{t}^2+\tilde{t}_1^2+\tilde{u}^2+\tilde{u}_1^2+M^2(\tilde{s}+\tilde{s}_1))(\tilde{t}x_2y_2+\tilde{t}_1x_1y_1-\tilde{u}x_2y_1-\tilde{u}_1x_1y_2)\Big],
\end{aligned} \tag{2.25}$$

with the following notation in use

$$\begin{aligned}
&D_1(a,b) = a^2 + b^2 + M^2(a+b)/\tilde{s}_1, \quad D_2(a,b) = a^2 + b^2 + M^2\tilde{s},\\
&\tilde{s} = p_1 \cdot p_2, \quad \tilde{s}_1 = q_1 \cdot q_2, \quad \tilde{t} = p_1 \cdot q_1, \quad \tilde{t}_1 = p_2 \cdot q_2,\\
&\tilde{u} = p_1 \cdot q_2, \quad \tilde{u}_1 = p_2 \cdot q_1, \quad x_i = k \cdot p_i, \quad y_i = k \cdot q_i.
\end{aligned} \tag{2.26}$$

The three A-functions, the phase space element $d\tau_3$ and all momenta are made dimensionless by using the beam energy E as a unit. This convention is kept in the following and in the program.

Before we continue the description of the M.C. algorithm let us define first carefully the kinematics of the hard bremsstrahlung events. For completness we include soft events in the discussion. We define the four momenta of all four fermions in terms of the Lorentz transformations connecting center of mass (CMS) laboratory system and the rest frame of the corresponding fermion. The advantage of this is that in the next section the same tranformations and reference frames are used to define spin states of these fermions. More precisely

$$\begin{aligned}
&p_1 = L_t(e^+)\hat{p}, \quad p_2 = L_t(e_-)\hat{p},\\
&q_1 = L_t(\tau^+)\hat{q}, \quad q_2 = L_t(\tau^-)\hat{q},\\
&k = p_1 + p_2 - q_1 - q_2,
\end{aligned} \tag{2.27}$$

where $\hat{p} = (m, 0, 0, 0)$ and $\hat{q} = (M, 0, 0, 0)$ are electron and τ four momenta in their rest frames. The Lorentz transformation L_t transform the corresponding momentum to CMS and $t = 0, 1$ discriminate between the soft and hard type of the event. The transformations read as follows

$$
\begin{aligned}
L_t(e^\pm) &= B_3(\pm\eta_e),\\
L_0(\tau^\pm) &= R_3(\phi)R_1(-\theta)B_3(\pm\eta_\tau),\\
L_1(\tau^\pm) &= R_3(\phi_1)R_1(-\theta_1)B_3(\zeta)R_3(\phi_2)R_1(-\theta_2)B_3(\pm\eta_\tau),
\end{aligned}
\tag{2.28}
$$

where

$$
\cosh\eta_e = 1/m,\ \cosh\eta_\tau = (1-k)^{1/2}/M,\ \cosh\zeta = (1-k)^{-1/2}(1-k/2).
$$

The transformation matrices

$$
R_1(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix}, \quad R_3(\phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
B_3(\eta) = \begin{pmatrix} \cosh\eta & 0 & 0 & \sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\eta & 0 & 0 & \cosh\eta \end{pmatrix} \tag{2.29}
$$

represent rotations around first and third axis and a boost along third axis. The transformations (2.23) are represented graphically in fig. 2. They link the $e^\pm$ and $\tau^\pm$ rest reference frames denoted in this figure as RS($e^\pm$) and RS($\tau^\pm$) with CMS. The other frames in this figure (CMS_1, CMS_γ, QMS and QMS_γ) are obtained by truncation of the chain of boosts and rotations in $L_t(\tau^\pm)$. Note that according to (2.27) the momentum of e^+ is directed along the third axis in CMS. The photon momentum is parallel to the third axis in CMS_γ and QMS_γ The systems QMS and QMS_γ are the rest systems of the τ pair. In spin amplitudes the polarization vectors of the photon will be defined in CMS_γ, the spin states of $\tau^\pm$ in RS($\tau^\pm$) and the spin states of $e^\pm$ in $\mathrm{RS}_1(e^\pm)$ which differ from RS($e^\pm$) by rotation $R_3(\phi)$ or $R_3(\phi_1)$, see fig. 2. The angles θ_1, ϕ_1 are the angular variables of the photon in CMS and the angles θ_2, ϕ_2 are the angular variables of τ^+ in QMS_γ.

We use the same set of five variables k, θ_i, ϕ_i to parametrise the phase element $d\tau_3$ in both branches of the hard part of the algorithm

$$\int d\tau_3 = \int\limits_{k_0}^{1-M^2} kdk \int\limits_0^{2\pi} d\phi_1 \int\limits_{-1}^{1} \beta dc_1 \int\limits_0^{2\pi} d\phi_2 \int\limits_{-1}^{1} \frac{1}{2} v(k) dc_2, \tag{2.30}$$

where $c_i = \cos\theta_i$, $\beta = (1-m^2)^{1/2}$. This is also the same set of the variables which was used in calculations the spin amplitudes in ref. [1].

Having defined the kinematics we return to the description of the algorithm. In the first branch of the hard part of the algorithm (fig. 1, p. 8-11) the four angles are generated according to matrix element A_{ini} representing bremsstrahlung from the initial state electrons. We now express A_{ini} in terms of our phase space variables

$$A_{ini} = \frac{q^2}{2k^2(1-k)\delta_1\delta_2}\Big[D_1(\tilde{t},\tilde{u})\left(1-\frac{m^2}{2(1-k)}\frac{\delta_1}{\delta_2}\right) + D_1(\tilde{t}_1,\tilde{u}_1)\left(1-\frac{m^2}{2(1-k)}\frac{\delta_2}{\delta_1}\right)\Big], \tag{2.31}$$

where

$$\begin{aligned} &D_1(\tilde{t},\tilde{u}) = 2\Delta(c_1,s_1,c_2,s_2) + 2M^2(1-k)^{-1}(1-k\delta_1/2)^2, \\ &D_1(\tilde{t}_1,\tilde{u}_1) = 2\Delta(-c_1,-s_1,c_2,s_2) + 2M^2(1-k)^{-1}(1-k\delta_2/2)^2, \\ &\delta_1 = 1-\beta c_1, \quad \delta_2 = 1+\beta c_1, \quad s_i = \sin\theta_i \end{aligned} \tag{2.32}$$

and $\quad \Delta(c_1,s_1,c_2,s_2) =$

$$(1-k(1-c_1)/2)^2 + ((1-k)^{1/2}s_1s_2\cos\phi_2 - (2-k)c_1c_2/2 - kc_2/2)^2.$$

It depends on c_2 and ϕ_2 rather mildly but it has sharp peaks coming from the factor $1/(\delta_1\delta_2)$. Using the inequalities

$$D_1(\tilde{t},\tilde{u}) \le 4(1-k\delta_1/2)^2, \quad D_1(\tilde{t}_1,\tilde{u}_1) \le 4(1-k\delta_2/2)^2 \tag{2.33}$$

and dropping some negative terms we find for A_{ini} the following DD function

$$\tilde{A}_{ini} = q^2\frac{2(1+(1-k)^2)}{k^2(1-k)}\frac{1}{\delta_1\delta_2}. \tag{2.34}$$

Since $\tilde{A}_{ini}$ depends only on c_1 we start WR procedure by generating variables c_2, ϕ_1 and ϕ_2 with constant probability and c_1 according to

$$\frac{2}{\delta_1\delta_2} = \frac{1}{\delta_1} + \frac{1}{\delta_2} = \frac{1}{1-\beta c_1} + \frac{1}{1+\beta c_1}. \tag{2.35}$$

It is done by generating c_1 according to $1/(1\text{-}\beta c_1)$ and then by random symmetrisation $c_1 \to -c_1$. The usual rejection with the weight

$$w_{ini} = A_{ini}/\tilde{A}_{ini} \tag{2.36}$$

is performed in order to reproduce the distribution A_{ini}. For a rejected event a new trial with new values of the c_1, c_2 and ϕ_2 variables is started leaving old k and ϕ_1, as shown in fig. 1. This is allowed because in the previous step k was generated according to the exact distribution and also because A_{ini} does not depend on ϕ_1.

The second branch in the hard part of the algorithm (fig. 1, p. 12-15) is very similar to the first one. The same set of the variables is used, only the role of c_1 and c_2 is interchanged. This can be seen from the formula

$$\begin{aligned} A_{fin} = \frac{q'^2}{2k^2\zeta_1\zeta_2}\Big[& D_2(\tilde{t},\tilde{u}_1)\left(1-\frac{M^2}{\zeta_2}\right) + D_2(\tilde{t}_1,\tilde{u})\left(1-\frac{M^2}{\zeta_1}\right) \\ & + \frac{1}{2}M^2k^2(\delta_1^2+\delta_2^2) - 8M^2k\Big] \end{aligned} \tag{2.37}$$

with $\quad \zeta_1 = 1 - v(k)c_2, \quad \zeta_2 = 1 + v(k)c_2,$

$$D_2(\tilde{t},\tilde{u}_1) = 2\Delta(-c_2,-s_2,-c_1,-s_1) + 2M^2,$$
$$D_2(\tilde{t}_1,\tilde{u}) = 2\Delta(c_2,s_2,-c_1,-s_1) + 2M^2,$$

where the main singularity is now in c_2 variable i.e. in the $1/(\zeta_1\zeta_2)$ factor. As in the previous case we find for A_{fin} a simple DD function

$$\tilde{A}_{fin} = q'^2\frac{2(1+(1-k)^2-M^2)}{k^2}\frac{1}{\zeta_1\zeta_2} \tag{2.38}$$

which depends only on c_2 variable. The variables ϕ_1 , c_1 and ϕ_2 are generated uniformly and c_2 according to $1/(\zeta_1\zeta_2)$. The weight

$$w_{fin} = A_{fin}/\tilde{A}_{fin} \tag{2.39}$$

is used for the rejection which reproduces the exact distribution A_{fin}.

Although the interference term A_{int} does not contribute to k-spectrum and to the integrated cross section, it introduces, however, some asymmetries in the phase space and it is imposed on the generated events in the last part of the hard part of the algorithm, see fig. 1, point 17, by additional WR procedure with the weight

$$w_{int}^h = \frac{1}{2}[1 + A_{int}/(A_{ini}+A_{fin})]. \tag{2.40}$$

3. Spin structure

In this Section it is explained how to include spin effects in the Monte Carlo simulation for arbitrary spin polarization of the electrons and for arbitrary decay of τ. It is done in two steps, see also fig. 1. First the $\tau^+\tau^-(\gamma)$ event is generated as described in Section 2, assuming that incoming electrons are unpolarized and that τ decay is not sensitive to its spin polarization. At this stage also the decay of each τ is simulated in its rest frame assuming that τ's are not polarized (fig.1, point 18). Then all spin effects from the polarization of the electrons and from sensitivity of the τ decay to its polarization are introduced in the single rejection using the weight w_{spin} (fig. 1, point 19).

The probability distribution (normalized to unity) of the decay of the polarized τ may be always written as

$$dp = (H^0 + \vec{H}\cdot\vec{w})d\Phi = H^0(1+\vec{h}\cdot\vec{w})d\tau_{dec}, \tag{3.1}$$

where $\vec{w}$ is the τ polarization vector in its rest frame and the integration element $d\tau_{dec}$ is the phase space element of all decay products or some other useful integration element, for example the integration element over momenta of all (or some) charged decay products only. H^0 is obviously the decay probability distribution $dp/d\tau_{dec}$ of the unpolarized τ. The vector $\vec{h} = \vec{H}/H^0$ is calculated from the matrix element of the decay process using standard spin projection operator $1+\gamma_5 \not{w}$ [8] for the decaying τ. Its magnitude $|\vec{h}|$ determines the efficiency of a given decay as a polarimeter and the direction of $\vec{h}$ gives the most probable direction of the τ polarization vector. Usually $\vec{h}$ is expressed in terms of the momenta of the decay products. In ref. [1] the list of vectors $\vec{h}$ is given for one prong dominant decays. The subprogram simulating these decays is also included in the test deck.

The multidifferential cross section for the production process $e^+e^- \to \tau^+\tau^-(\gamma)$ and the subsequent decay of both τ's is given by the following formula [1]

$$\begin{aligned} d\sigma &= \sum_{a,b,c,d=0}^{3} \tilde{\epsilon}^a_{(1)}\tilde{\epsilon}^b_{(2)} R_{abcd} H^c_{(1)} H^d_{(2)} d\tau_{prod} d\tau^{(1)}_{dec} d\tau^{(2)}_{dec}, \\ &= R_{0000} H^0_{(1)} H^0_{(2)} \sum_{a,b,c,d=0}^{3} \tilde{\epsilon}^a_{(1)}\tilde{\epsilon}^b_{(2)} r_{abcd} h^c_{(1)} h^d_{(2)} d\tau_{prod} d\tau^{(1)}_{dec} d\tau^{(2)}_{dec}, \end{aligned} \tag{3.2}$$

where $d\tau_{prod} = d\tau_2$ for $k < k_0$ and $d\tau_{prod} = d\tau_3$ for $k > k_0$. The four-vectors $\tilde{\epsilon}^a_{(k)}$ with $k = 1, 2$ denoting e^+ and e^- are defined in the rest frame of the corresponding electron as $(1, \vec{\epsilon}^{(k)})$ where $\vec{\epsilon}^{(k)}$ stands for the standard spin polarization vector of e^+ $(k = 1)$ and e^- $(k = 2)$. Similarily, vectors $h^a_{(k)}$ are defined in the rest frame of τ^+ $(k = 1)$ and of τ^- $(k = 2)$ as $(1, \vec{h}_{(k)})$.

We want to stress that the four vectors $\tilde{\epsilon}_{(k)}$ and $h_{(k)}$ are defined and used only in the corresponding rest systems of $e^\pm$ and $\tau^\pm$. Their unit zero components are introduced to make formula (3.2) more compact.

The tensor R_{abcd} in eq. (3.2) can be calculated out of spin amplitudes for the production process $e^+e^- \to \tau^+\tau^-(\gamma)$. Inserting $\vec{\epsilon}^{(k)} = 0$ and $\vec{h}^{(k)} = 0$ into eq.(3.2) and integrating it over decay phase spase we see that

$$R_{0000} = d\sigma / d\tau_{prod} \tag{3.3}$$

is simply the differential cross section used in Section 2 to simulate the production process in the spinless case. We choose the multidifferential cross section

$$d\tilde{\sigma} / d\tau_{prod} d\tau^{(1)}_{dec} d\tau^{(2)}_{dec} = l_{sp} R_{0000} H^0_{(1)} H^0_{(2)} \tag{3.4}$$

as the DD distribution for the combined τ production and τ decay process. The resulting spin weight, used in the final WRT,

$$w_{spin} = \frac{d\sigma}{d\tilde{\sigma}} = \frac{1}{l_{sp}} \sum_{abcd/0}^{3} \tilde{\epsilon}^a_{(1)} \tilde{\epsilon}^b_{(2)} r_{abcd} h^c_{(1)} h^d_{(2)} \tag{3.5}$$

is simply equal to the ratio of the desired, spin dependent, distribution (3.2) to the spinless (averaged over the decay probability) distribution from eq. (3.4). The factor l_{sp} is chosen empirically such that $w_{spin} \leq 1$. For unpolarized $e^\pm$ it is enough to set $l_{sp} = 2$, but for arbitrary beam polarizations l_{sp} must be set to 8.

The correlation tensor $r_{abcd}{=}R_{abcd}/R_{0000}$, includes all information about spin effects in the production process. For example the dependence of the differential cross section on electron polarizations is given by $\sum \epsilon^a_{(1)} \epsilon^b_{(2)} r_{ab00}$, the correlation tensor r_{ab} used in ref. [1] is equal to r_{00ab}, etc..

In the remainder of this Section we show how the correlation tensor r_{abcd} and the spin weight w_{spin} is obtained out of known spin amplitudes $\mathcal{M}^\beta_{\lambda_1, \lambda_2, \alpha_1, \alpha_2}$ for the production process $e^+e^- \to \tau^+\tau^-(\gamma)$. These spin amplitudes were calculated in ref. [1] with the helicity-like choice of the spin states; For τ leptons α_1 and α_2 are spin projections of τ^+ and τ^- on the third axis in the corresponding rest frame RS($\tau^\pm$) defined in Section 2, see also fig. 2. For incoming $e^\pm$ the indices λ_1 and λ_2 are spin projections of e^+ and of e^- on the third axis in the corresponding rest frame $\mathrm{RS}_1(e^\pm)$ defined in Section 2 and in fig. 2. The index $\beta = 1,2$ numbers the two transverse polarizations of the photon ϵ_β. They are choosen to be the unit vectors along the first and second axis in CMSγ, ie. $(\epsilon_\beta)^\mu = \delta^\mu_\beta$ in this frame. The user supplied polarisation vectors of $e^\pm$ are supposed to be provided in RS($e^\pm$) frames related to CMS by a pure z-boost. They differ from $\mathrm{RS}_1(e^\pm)$ by rotation $R_3(\phi)$ or $R_3(\phi_1)$ depending on the type of event, see fig. 2. For the radiatively corrected τ production process it was found [1] that the spin amplitudes T^β_{ij} related to $\mathcal{M}$-amplitudes through the linear relation

$$\mathcal{M}^\beta_{\lambda_1,\lambda_2,\alpha_1,\alpha_2} = \sum_{ij=1}^{4} T^\beta_{ij} b^i_{\lambda_1\lambda_2} b^j_{\alpha_1\alpha_2}, \tag{3.6}$$

where

$$\begin{aligned} b^1_{\lambda_1\lambda_2} &= 2\lambda_1\delta_{\lambda_1,\lambda_2}, \qquad & b^2_{\lambda_1\lambda_2} &= \delta_{\lambda_1,\lambda_2}, \\ b^3_{\lambda_1\lambda_2} &= 2\lambda_1\delta_{\lambda_1,-\lambda_2}, \qquad & b^4_{\lambda_1\lambda_2} &= \delta_{\lambda_1,-\lambda_2}, \end{aligned} \tag{3.7}$$

are four times shorter than $\mathcal{M}$-amplitudes and therefore we use the in the program. The tensor R_{abcd} is given in terms of them by the following expression

$$R_{abcd} = \sum_{ijkl\beta} E^*_{abij} T^\beta_{ik} (T^\beta_{jl})^* E_{cdkl}, \tag{3.8}$$

where

$$E_{abij} = \frac{1}{2} \sum_{\alpha_1\alpha_2\bar\alpha_1\bar\alpha_2=\pm1/2} b^i_{\alpha_1\alpha_2} b^j_{\bar\alpha_1\bar\alpha_2} \sigma^a_{\bar\alpha_1\alpha_1} \sigma^b_{\bar\alpha_2\alpha_2}$$

and the matrices σ^k, $k = 1,2,3$ are the standard Pauli matrices supplemented with $\sigma^0 = I$. In the numerical evaluation of w_{spin} we find it more efficient to contract first the vectors $\tilde\epsilon$ and h with the tensor E and then to contract the resulting tensor with the amplitudes T^β_{ij}. The following expression for the spin weight is used in the program

$$w_{spin} = l_{sp}^{-1} \sum_{ijkl\beta} u^*_{ij} T^\beta_{ik} (T^\beta_{jl})^* v_{kl} \Bigg/ \sum_{ij\beta} |T^\beta_{ij}|^2 , \tag{3.9}$$

where

$$
\begin{aligned}
v_{kl} &= \sum_{cd} E_{cdkl} h^c_{(1)} h^d_{(2)}, \\
u_{ij} &= \sum_{ab} E_{abij} \tilde{\epsilon}^a_{(1)} \tilde{\epsilon}^b_{(2)}.
\end{aligned}
\tag{3.10}
$$

The explicit formulae (used in the program) for v_{ij} elements define also all nonzero elements out of 256 elements of the tensor E_{abij}

$$
\begin{aligned}
v_{11} &= (00) - (11) + (22) + (33), \\
v_{22} &= (00) + (11) - (22) + (33), \\
v_{33} &= (00) - (11) - (22) - (33), \\
v_{44} &= (00) + (11) + (22) - (33), \\
v_{12} &= (30) + (03) + i(12) + i(21), \quad v_{21} = v^*_{12}, \\
v_{34} &= (30) - (03) - i(12) + i(21), \quad v_{43} = v^*_{34}, \\
v_{13} &= (01) - (10) - i(23) + i(32), \quad v_{31} = v^*_{13}, \\
v_{24} &= (01) + (10) + i(23) + i(32), \quad v_{42} = v^*_{24}, \\
v_{14} &= (31) + (13) + i(02) + i(20), \quad v_{41} = v^*_{14}, \\
v_{23} &= (31) - (13) + i(02) - i(20), \quad v_{32} = v^*_{23},
\end{aligned}
\tag{3.11}
$$

where the following shorthand notation was used

$$(cd) = h^c_{(1)} h^d_{(2)}.$$

Note also that $v_{ij} = \delta_{ij}$ for $\vec{h}_{(1)} = \vec{h}_{(2)} = 0$.

We want to comment now on the question in which rest reference frame (with what axes exactly) the vectors $\vec{\epsilon}^{(k)}$ and $\vec{h}^{(k)}$ should be taken for the substitution into the equations (3.2) and (3.9). These vectors should be provided in the same well defined reference frames of electrons and τ's which were originally used to quantize their spins in the calculations of the spin amplitudes ie. vectors $\vec{h}^{(k)}$ should be provided in the corresponding RS($\tau^\pm$) frame and vectors $\vec{\epsilon}^{(k)}$ in the corresponding RS$_1$($e^\pm$) frame. From this we see that the initial state polarization vectors $\vec{\epsilon}^{(k)}$ have to be rotated using $R_3(\phi)$ (for $k < k_0$) or $R_3(\phi_1)$ (for $k > k_0$) around the beam axis before they are used in the formula (3.10) due to the difference between RS($e^\pm$) and RS$_1$($e^\pm$) frames. The decay of each $\tau^\pm$ is assumed to take place in the same corresponding rest frame $RS(\tau^{\pm})$ where the vectors $\vec{h}_{(k)}$ were defined. The user of the program, however, does not need to worry about the definitions of these two $\tau^{\pm}$ rest frames because a user routine is supplied together with the program which transforms any four vector (momentum of the decay product) from these two frames to the laboratory system CMS using the transformation $L_t(\tau^{\pm})$. The rotation of the (user supplied) vectors $\vec{e}_{(k)}$ from RS($e^{\pm}$) to RS$_1$($e^{\pm}$) is also done automatically inside the program.

To construct the spin weight for soft event we need four sets of 16 spin amplitudes i.e. for the lowest order process, for contributions from the box diagrams, from magnetic part of the vertex correction and from Z_0 exchange. The nonzero spin amplitudes for the lowest order process are

$$T^0_{11} = 1, \qquad T^0_{22} = c, \qquad T^0_{24} = iMs, \tag{3.12}$$

for the magnetic part of the final state vertex correction are

$$T^M_{11} = \tilde{F}_2, \quad T^M_{22} = c\tilde{F}_2, \quad T^M_{24} = is\tilde{F}_2/M, \tag{3.13}$$

for the infrared finite part of the sum of the two box diagrams are

$$T^B_{11} = qq'\frac{\alpha}{2\pi}Z_1, \quad T^B_{22} = qq'\frac{\alpha}{2\pi}Z_2, \quad T^B_{24} = qq'\frac{\alpha}{2\pi}iMZ_3 \tag{3.14}$$

and for the contribution from the electroweak boson Z_0 exchange are

$$\begin{aligned} T^Z_{11} &= (qq')^{-1}\chi(v\tilde{v} + c\, a\tilde{a}), \quad T^Z_{12} = (qq')^{-1}\chi(v\tilde{a} + c\, a\tilde{v}), \\ T^Z_{21} &= (qq')^{-1}\chi(a\tilde{v} + c\, v\tilde{a}), \quad T^Z_{22} = (qq')^{-1}\chi(a\tilde{a} + c\, v\tilde{v}), \end{aligned} \tag{3.15}$$

where $c = \cos\theta$ and $s = \sin\theta$. The following notation and functions were introduced in eqs. (3.12) to (3.15): $\tilde{F}_2$ is the magnetic part of the final state vertex correction defined in Section 2. The three functions Z_i in the spin amplitudes from two box diagrams

$$\begin{aligned} Z_1 &= -i(X_1(c) - X_1(-c)), \\ Z_2 &= -i(X_2(c) + X_2(-c)), \\ Z_3 &= -i(X_3(c) + X_3(-c)), \end{aligned} \tag{3.16}$$

contain the three functions X_i which were calculated in ref. [1]

$$
\begin{aligned}
-iX_k = \sum_{j=1}^{6} c_{kj} A_j &= c_{k1}\tilde{A} + c_{k2}(\tilde{B} + i\pi\tilde{A}) \\
&+ c_{k3}(-iF'_\Delta) + c_{k4}(-iF'_Q) + c_{k5}(\ln\frac{4}{m^2} + \ln\frac{4}{M^2} + 2\pi i) \\
&+ c_{k6}(\ln\frac{4}{m^2} - \ln\frac{4}{M^2}).
\end{aligned}
\tag{3.17}
$$

The coefficients $c_{kj}(c)$ are listed in table 1 and the four complex functions

$$
\begin{aligned}
\tilde{A} &= 2\ln\frac{1-\beta' c}{2} + \ln\frac{4}{m^2} + \ln\frac{4}{M^2}, \\
\tilde{B} &= \ln^2\frac{1-\beta' c}{2} - 2\mathrm{Li}_2\Big(\frac{2-2\beta' c - M^2}{2(1-\beta' c)}\Big) - \frac{1}{2}\ln^2\frac{4}{m^2} - \frac{1}{2}\ln^2\frac{4}{M^2}, \\
-iF'_\Delta &= \frac{1}{2}\ln^2\frac{4}{m^2} + \frac{\pi^2}{6} - i\pi\ln\frac{4}{m^2}, \\
-iF'_Q &= \frac{1}{\beta'}\Big[\frac{1}{2}\ln^2\frac{1+\beta'}{1-\beta'} + 2\mathrm{Li}_2\Big(\frac{-1+\beta'}{1+\beta'}\Big) + \frac{\pi^2}{6} - i\pi\ln\frac{1+\beta'}{1-\beta'}\Big]
\end{aligned}
\tag{3.18}
$$

were introduced in ref. [2]. In the above formulae and in the following we omit from T_{ik} a common complex normalization constant irrelevant for calculation of w_{spin}.

Table 1

The coefficients c_{ik} in the formula (3.17) for functions Z_i in the box amplitudes. The following notation is used: $c = \cos\theta,\ s = \sin\theta,\ \omega = 2 - 2\beta' c - M^2$.

The spin weight for soft event is finally calculated in the program using the formula

$$w_{spin}^{soft} = \frac{1}{l_{sp}} \frac{S(T^0, T^1)}{S_0(T^0, T^1)} \tag{3.19}$$

where

$$\begin{aligned}
S(T^0,T^1) &= \frac{1}{2} \sum_{ijkl=1}^{4} \left(u_{ij}^* T_{ik}^0 (T_{jl}^1)^* v_{kl} + u_{ij}^* T_{ik}^1 (T_{jl}^0)^* v_{kl} \right), \\
S_0(T^0,T^1) &= \frac{1}{2} \sum_{ik=1}^{4} \left(T_{ik}^0 (T_{ik}^1)^* + T_{ik}^1 (T_{ik}^0)^* \right), \\
T_{ik}^1 &= T_{ik}^0 (1+\delta_t) + 2(T_{ik}^M + T_{ik}^B + T_{ik}^Z), \\
\delta_t &= \delta^{SX} - 2\mathrm{Re}\Pi + qq' \frac{\alpha}{\pi} \delta_{BREM}(c).
\end{aligned} \tag{3.20}$$

The term proportional to $1+\delta_t$ collects all contributions which have spin amplitudes proportional to the lowest order amplitudes and the other terms represent the interference of the box, $\tilde{F}_2$ and Z_0 amplitudes with the lowest order amplitudes. The formula (3.19) is more complicated than that of eq. (3.8) due to omission of $O(\alpha^4)$ terms from the tensor R_{abcd}.

We calculate the spin weight for the hard photon bremsstrahlung process from 32 spin amplitudes which read

$$T_{ik}^{\beta} = [y\bar{\beta}(m^2+s_1^2)]^{-1} q t_{ik}^{\beta} + \ [\bar{\beta}(\tilde{M}^2 + v^2(k) s_2^2]^{-1} q' t_{ik}^{\prime\beta} \tag{3.21}$$

where

$$y = (1-k)^{1/2}, \quad \tilde{M} = M/y, \quad \bar{\beta} = k/(2y), \quad v(k) = (1-\tilde{M}^2)^{1/2}.$$

The two tensors t and t' are related to bremsstrahlung from the initial and final state fermions and are given by

$$\begin{array}{lll}
t_{ik}^{\beta} = s_1 f_{ik}^{\beta}, & t_{ik}^{\prime\beta} = g_{ik}^{\beta}, & \text{for} \quad i=1,2, \\
t_{ik}^{\beta} = -m\xi^{\beta} f_{i-2,k}^{\beta}, & t_{ik}^{\prime\beta} = 0, & \text{for} \quad i=3,4, \\
\xi^1 = c_1, & \xi^2 = -c_1 \bar{\beta}/\bar{\gamma}, & \bar{\gamma} = (1+\bar{\beta}^2)^{1/2}.
\end{array} \tag{3.22}$$

where the elements of f_{ik}^{β} and g_{ik}^{β} are listed in table 2. The tensor g_{ik}^{β} from table 2 has to be rotated in the photon polarization index β

$$\begin{aligned}
g_{ik}^1 &= \cos\phi_2 g_{ik}^{1'} - \sin\phi_2 g_{ik}^{2'}, \\
g_{ik}^2 &= \sin\phi_2 g_{ik}^{1'} + \cos\phi_2 g_{ik}^{2'},
\end{aligned} \tag{3.23}$$

before it can be substituted into eq. (3.21) because in table 2 it is given with a different choice of the photon polarization than the tensor f_{ik}^{β} (see also ref. [1]). Finally the spin weight is calculated using the formula

$$w_{spin}^{hard} = \frac{1}{l_{sp}} \frac{S(T,T)}{S_0(T,T)}, \tag{3.24}$$

where the summation over $\beta = 1, 2$ is also included in the sums S and S_0.

The total cross section depends on the polarization of the initial particles and is calculated using the average weight from the generation of the photon momentum and from the average spin weight

$$\sigma_{tot} = \sigma_p l_{sp} < w_{spin} > (\rho_s + < w_k > \int_{k_0}^{1-M^2} \tilde{\rho}(k) dk). \tag{3.25}$$

The average spin weight $< w_{spin} >$ is also averaged over the photon spectrum.

Table 2

The list of element f^{β}_{ik} and g^{β}_{ik} in the hard bremsstrahlung amplitudes. The following notation is used: $c_{\phi} = \cos\phi_2$, $s_{\phi} = \sin\phi_2$, $\bar{\beta} = \cosh\zeta$, $\bar{\gamma} = \sinh\zeta$, $\tilde{M} = (1 - v^2)^{1/2}$, $v = v(k)$.

4. Subprograms, common blocks, variables

In this Section we give more details on the program and its use. The global structure of the program as organised in the test deck is shown in fig. 3. In table 3 we list all subprograms together with a brief description of their purpose and parameters. Here we shall give more details on subprograms directly relevant to the user and we comment on possible modifications of the program. The list of all common blocks and of all variables in the common blocks is given in table 4.

Table 3

The list of subprograms.

START(ENE,SINW2,INRAN)

This routine must be called by the user before first event is generated. It initialises some variables stored in several common blocks and used in other routines. All parameters are set by the user. ENE is the center of mass system beam energy $\sqrt{s}/2$ in GeV units. SINW2 is $\sin^2\theta_W$ of the standard electroweak model [9] and INRAN is positive nonzero integer used to initialise the random number generator RNF100. Radiative corrections may be switched off for tests by seting ENE¡0, program uses absolute value of ENE anyway. Similarly the Z_0 exchange may be switched off by seting SINW2=-0.5.

KORAL(E1,E2)

This routine is called to generate a single M.C. event. The two vectors E1, E2 must be defined by the user, they are the polarisation vectors $\vec{\epsilon}_{(1)}$ and $\vec{\epsilon}_{(2)}$ of the positron and electron in their rest frames RS($e^{\pm}$) related to CMS by pure z-boost, see fig. 2.

FINISH(CSTCM)

This routine may be called by the user in order to calculate the total cross section CSTCM=σ_{tot}, as defined in eq. (3.25) in units of cm^2. It should be called after the last event was generated. It also prints some information on the average weights the statistical error of σ_{tot}.

TRALOR(KTO,VEC)

It transforms the four momentum VEC of any τ decay product from τ^+ rest system RS(τ^+), KTO=1, or from τ^- rest frame RS(τ^-), KTO=2, to the center of mass (laboratory) system CMS. As for other four vectors in the program first three components of VEC are momenta components and VEC(4) is the energy.

DECAY(KTO,H)

This subrogram simulates the decay of τ^+ or τ^- for KTO=1 and KTO=2 in the corresponding τ rest system RS($\tau^\pm$). It defines vector H(i)=$h^i_{(KTO)}$, $i = 1,2,3$, H(4)=1, which is used in KORAL to calculate the spin weight. It simulates one of the three types of the one prong decays $\tau^\pm \to e^\pm, \rho^\pm, \pi^\pm$, depending on the value of the control variables in the common block JAK, see table 4.

EVENT

This routine generates spinless events as described in section 2. It defines angular variables in common block UTIL1 and the four momenta of $\tau^\pm$ and γ in common block UTIL.

AMPLIT(T1,T2)

It calculates complex spin amplitudes. For soft events T1 and T2 are T^0_{ik} and T^1_{ik} in eq. (3.19) and for hard events they are T^1_{ik} and T^2_{ik} of eq. (3.21).

SPIN(V,E,F)

It calculates the tensor v_{ik} (or u_{ik}) denoted as V out of two four vectors E and F according to eqs. (3.11).

BRMBOX

This routine calculates the soft bremsstrahlung factor $\delta_{BREM}(c)$ and the box contribution $\delta_{BOX}(c)$ used in the soft photon differential cross section (2.20) storing them in the common block SIGMA. It defines also three functions Z_i used in the spin amplitudes (3.14), see also common block BOXY.

WINTH(DUMMY)

This function with dummy argument returns the value of the weight (2.40) used to introduce the interference of the initial state and final state hard bremsstrahlug.

RRGEN(R)

This function generates photon momentum k according to the density distribution $\tilde{\rho}(k)$. It is taken from ref. [5]. R is the uniform random number in the range (0,1).

XKDEF(XK)

It returns the value of $\int_0^{XK} \tilde{\rho}(k)dk$. It is called in RRGEN.

DILOG(X)

It calculates the dilogarithm function $\mathrm{Li}_2(x)$ [10].

SETWS(IDF,SIN2,QC,CV,CA,AMZ,GAMM)

This routine, taken from ref. [5] calculates Z_0 coupling constants v=CV, a=CA, and Z_0 mass AMZ out of $\sin^2\theta_W$=SIN2. IDF is the fermion identifier. See also comments inside this routine.

PIRET(S)

This function returns the value of the real part of the vacuum polarisation function $\mathrm{Re}\Pi(s)$, where s=S is given in GeV^2 units. The hadronic contribution is calculated according to ref. [11] using smooth logarithmic parametrization. Leptonic contributions are also included. Near the τ threshold a better approximation, including resonance structure in the hadronic part should be used, see ref. [11].

RNF100(IGEN)

Multiple random number generator with shuffling [12].

DECTAU(JAKI,HH,PP)

Subprogram called in DECAY to simulate one prong decay of one of τ's. JAKI is the identifier of the decay type, see also common block JAK, HH is the vector $\vec{h}$ and PP is the four momentum of the charged decay product in the τ rest frame.

RNDM

Simple uniform random number generator used in many subprograms.

RNDP

Simple uniform random number generator called only in DECTAU.

SPHERA(X)

Subprogram generating uniformly a three dimensional vector on the unit sphere. It is called in DECTAU only.

RENOR

It normalizes the content of the histogram.

Table 4

The list of common blocks and variables in them.

Common	Variab.	Definition
CONST	PI	π=3.141593...,
	ALFA	α, the fine structure constant, $\alpha^{-1} = 137.036...$,
	ALF1	α/π,
	QE2	q^2,
	QF2	q'^2,
	QEF	qq'.
UTIL	QP(4)	The four vector of τ^+ in CMS (which has third axis directed along the momentum of e^+), $QP(i) = q_1^i$ for $i = 1, 2, 3$ and $QP(4) = q_1^0$, all components are in beam energy units , the same convention is used for other four vectors,
	QM(4)	q_2, the four momentum of τ^-,
	PH(4)	k, the four momentum of the photon, it is set to zero for $k < k_0$.
DECPP	PP1(4)	Four momentum of the τ^+ decay product in RS(τ^+),
	PP2(4)	four momentum of the τ^- decay product in RS(τ^-) .
UTIL1	XK	k, the energy of the photon in units of the beam energy,
	C1,S1	$\cos\theta_1,\ \sin\theta_1$ or $\cos\theta,\ \sin\theta$,
	C2,S2	$\cos\theta_2,\ \sin\theta_2$,
	CF,SF	$\cos\phi_2,\ \sin\phi_2$,
	CG,SG	$\cos\phi_1,\ \sin\phi_1$ or $\cos\phi,\ \sin\phi$,
	V	$v(k)$.
UTIL2	XK0	k_0, the parameter defining soft radiation, set to 0.01 in START,
	XKMIN	minimum k for hard radiation, set to k_0 in START
	XKMAX	maximum k for hard events, set to $1 - M^2$ in START
ENERG	ENE	$E = \sqrt{s}/2$, beam energy (GeV),
	AEL2	$m^2 = m_e^2/E^2$,
	AMF2	$M^2 = m_\tau^2/E^2$,

	AMF	$M = m_\tau/E$,
	ALGEL	$\ln(4/m^2)$,
	ALGMF	$\ln(4/M^2)$,
	BETI	$\beta_{ini} = 2q^2\alpha(ln(4/m^2) - 1)/\pi$,
	BT1	$\beta' = (1 - M^2)^{1/2}$,
	ATH2	$q'^2 \ln((1 + \beta')/(1 - \beta'))$.
SIGMA	SIG0	$\sigma_p = (qq')^2\pi\alpha^2/3E^2$ (GeV^{-2}),
	SIG0CM	σ_p in cm^2 units,
	SOFT	$\delta^{SX} - 2Re\Pi + \delta^Z$,
	SOF1	$\delta^{SX} - 2Re\Pi$,
	F2RE	$Re\tilde{F}_2$,
	BRM	$\delta_{BREM}(c)$,
	BOX	$\delta_{BOX}(c)$.
BOXY	Z1,Z2, Z3	The three complex functions Z_i, $i = 1, 2, 3$ in the box spin amplitudes,
WEAK	QCE	q,
	QCF	q',
	CVE,CVF	v, $\tilde{v}$,
	CAE,CAF	a, $\tilde{a}$,
	ZPROP	χ.
INOUT	NINP	Input unit number, not used,
	NOUT	Output unit number, defined in START.
CONTRL	SWT(6)	It contains the number of trials, the sums of weights, and sums of squared weights used to calculate ¡w_k¿ and ¡w_{spin}¿,
	KSPIN	control variable set to 1 in START, if set to 0 then spin effects are switched off (for tests).
RNF	INIRAN	Initialisation constant for RNF100 generator,
	NGEN	the number of parallel subgenerators in RNF100 random number generator, set to 9 in START.

JAK	JAK1	This variable determines the type of the one prong decay of τ^+; JAK1$= 1, 2, 3$ denotes τ^+ decaying into e^+ (or μ^+), ρ^+ and π^+ respectively,
	JAK2	the analogous variable JAK2=1,2,3 for τ^-.

The user who wants to generate a series of the M.C. events must call three subprograms START, KORAL and TRALOR. He may call also (optionally) another subprogram FINISH. The order of calling to these routines must remain exactly as in the presented test deck. The initialisation subprogram START must be called before any event is generated. Through its parameters the user sets the beam energy, the electroweak mixing angle, and the initialisation constant for the random number generator. The type of the (one prong) τ decay is choosen by the user either once for all or separately for each event through control variables in the common block JAK, see table 4.

In order to produce a M.C. event, the subprogram KORAL is called. The electron spin polarization vectors must be supplied as its parameters. For unpolarized electrons (polarization vectors set to zero) the program is faster by a factor of ten on the average. The user has access to the four momenta of the τ leptons and of the photon through the common block UTIL, see table 3. They are defined in CMS (laboratory) reference frame which, according to definitions in Section 2, has its third axis directed along e^+ momentum. The momenta of the charged decay products are placed in the common block DECPP. These momenta are provided by the subprogram DECAY (called in KORAL) in the rest frames of the corresponding τ lepton RS($\tau^\pm$). The user may easily transform them to the CMS using the subprogram TRALOR, as it is done in the main program of the test deck. This transformation is done outside DECAY which simulates τ decay, because the DECAY might be called in KORAL a few times before an event with proper spin properties is accepted.

The exact integrated cross section σ_{tot} which includes all corrections and effects defined in formula (3.25), may be obtained by calling FINISH after the last event in the M.C. series was generated. Since this cross section is calculated using some average weights from the generation, it has a statistical error which is quite small for a resonably large number ($\sim 10^4$) of the generated events.

It should be stressed that the program has some autonomous parts which may be used separately or may be replaced with the user supplied segments. One example is the subprogram DECAY which simulates one prong decays of τ in its rest frame. It communicates with the rest of the program only through its parameters and may be replaced by the user with his own subprogram simulating any τ decays. In fact DECAY is treated as a subprogram which is in principle supplied by the user. The version of DECAY included in this test deck is rather simple and we refer the reader to the comments inside DECAY and related subprograms for more details on its structure. If the user provides his own subprogram DECAY simulating the τ decay in its rest frame then he is still strongly recommended to use TRALOR to transform all decay products from τ rest frame to the CMS. The user must be aware of the fact that, by using another routine to do this transformation (not identical with TRALOR), he may be unable to obtain correct spin correlations. Spin effects in the final state are introduced in KORAL using $\vec{h}$-vectors provided by DECAY. The user supplied DECAY routine should also provide this vector in order to reproduce correctly the spin effects in the final state. If the decay is weakly sensitive to τ polarization or the user wants to neglect these effects then he may set $\vec{h} = 0$.

Another example is the subprogram EVENT which, together with subprograms called in EVENT, can be used to simulate the production of the heavy quark or lepton, neglecting spin effects (unimportant due to quark fragmentation), but still keeping radiative corrections and mass effects. For example to simulate the production of the quark b the user should replace in START the τ mass with the b mass and the final fermion identifier IDFIN (the argument of SETWS in START) with IDFIN$= -4$. The colour factor is then automaticaly included. After calling EVENT the user may treat QP and QM in common UTIL as the momenta of $\bar{b}$ and b.

The subprograms EVENT and DECAY are both called in KORAL where all spin effects are introduced by the rejection technique. In order to speed up the program the case of unpolarized beams is isolated in KORAL in the separate part. In this case the average spin weight is equal 1/2 exactly and furthermore one does not need to repeat the calling of EVENT for rejected events. This option is marked with a broken line in fig. 1. For polarized beams calling EVENT is repeated for the rejected events and the rejection rate depends strongly on the longitudinal components of e^+ and e^- similarly as the total cross section itself.

Due to the short word of the IBM installation, all floating point and complex variables in this version of the program are in double precision. It is done, however, by use of the IMPLICIT and GENERIC statements, therefore it is rather simple to produce a version in single precision, if necessary.

The possibility of switching off the radiative corrections, spin effects and Z_0 exchange (in any combination) is provided in the program. For more details see the description of common block CONTRL in table 4 and of START parameters in table 3.

5. Test run

In our test run we generate 20000 events at a beam energy E=22 GeV. The leptonic decay

$$\tau^{\pm}(q_i) \rightarrow e^{\pm}(r_{\pm}) + \nu\bar{\nu}$$

is chosen. The coupling constants of Z_0 and its mass are taken from the standard electroweak model [9] with $\sin^2\theta_W = 0.23$. The incoming electron and positron have polarization vectors transverse to the beam axis and opposite to each other, $\vec{\epsilon}_{(1)} = (1,0,0)$, $\vec{\epsilon}_{(2)} = (-1,0,0)$. The main program calls all four user routines START, KORAL, TRALOR and FINISH demonstrating their use. It generates a series of the M.C. events and produces three histograms.

The printout starts with an output printed by START and with the polarization vectors of e^+ and e^-. START prints its parameters and some information on the cross sections and initialised variables. Next come three tests A, B and C and the output from FINISH.

Test A gives a list of the four-momenta of τ^+, τ^- and photon , together with the momenta of the outgoing e^+ and e^- decay products. All these momenta are printed for the first generated event. Then comes an output from FINISH which gives the value of the total integrated cross section σ_{tot} with its error and related information.

Test B consist of three histograms. The first histogram is the distribution $d\sigma/\zeta$ (normalized in nanobarns/radians) where ζ is the acollinearity angle between the momenta $\vec{r}_+$ and $-\vec{r}_-$ of the decay products. The next two histograms contain the angular distribution $d\sigma/d\cos\theta_+$ (in units of nanobarns) where the angle θ_+

is the polar angle of the outgoing e^+. The azimuthal angle ϕ_+ of this particle is limited in the first of these two histograms to the region $-45^o < \phi_+ < 45^o$ and $135^o < \phi_+ < 225^o$ while in the second histogram ϕ_+ should be outside of this range. The acollinearity cutoff $\zeta < 30^o$ is used in both angular distributions. The clear difference between these two distributions is due to the transverse polarization of the initial beam particles.

In this part of the test deck we show how to use the total cross section supplied by START in order to normalize properly the histograms. Histograms are produced using the CERN histogramming package HBOOK [13]. Histograms are normalized in the subroutine RENOR. If HBOOK is not available on some installation then the routines HBOOK1, HFILL, HPRINT and RENOR may be replaced in the first run with dummy subprograms which contain only the RETURN statement.

Test C lists the angular charge asymmetry calculated from the angular distributions as defined in the second and third histogram.

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Figure captions

1. The scheme of the Monte Carlo algorithm. Rectangular boxes denote generation of the variables and the romboid boxes mark decision points, r denotes current uniform random number in the (0,1) range. The return along the broken line takes place in the case of unpolarized $e^{\pm}$.

2. The scheme of the reference systems used in the program and in the definitions of the spin states: a) for soft events and b) for hard events. The reference systems are represented by boxes and the Lorentz transformations relating these systems are marked on the links between the boxes.

3. The global structure of the program. The simple random number generators called in many subprograms are omitted in the scheme.

Figure 2

$$
\begin{tikzcd}
\boxed{\mathrm{RS}(\tau^+)} \arrow[r, "B_3(\eta_\tau)"] & \boxed{\mathrm{QMS}} \arrow[d, dash, "R_1(-\theta)"] & \boxed{\mathrm{RS}(\tau^-)} \arrow[l, "B_3(-\eta_\tau)"'] \\
\boxed{\mathrm{RS}_1(e^+)} \arrow[r, "B_3(\eta_e)"] \arrow[d, dash, "R_3(\phi)"] & \boxed{\mathrm{CMS}_1} \arrow[d, dash, "R_3(\phi)"] & \boxed{\mathrm{RS}_1(e^-)} \arrow[l, "B_3(-\eta_e)"'] \arrow[d, dash, "R_3(\phi)"] \\
\boxed{\mathrm{RS}(e^+)} \arrow[r, "B_3(\eta_e)"] & \boxed{\mathrm{CMS}} & \boxed{\mathrm{RS}(e^-)} \arrow[l, "B_3(-\eta_e)"']
\end{tikzcd}
$$

(2a)

$$
\begin{tikzcd}
\boxed{\mathrm{RS}(\tau^+)} \arrow[r, "B_3(\eta_\tau)"] & \boxed{\mathrm{QMS}} \arrow[d, dash, "R_1(-\theta_2)"] & \boxed{\mathrm{RS}(\tau^-)} \arrow[l, "B_3(-\eta_\tau)"'] \\
& \boxed{\mathrm{QMS}_\gamma} \arrow[d, dash, "B_3(\zeta)R_3(\phi)"] & \\
& \boxed{\mathrm{CMS}_\gamma} \arrow[d, dash, "R_1(-\theta_1)"] & \\
\boxed{\mathrm{RS}_1(e^+)} \arrow[r, "B_3(\eta_e)"] \arrow[d, dash, "R_3(\phi_1)"] & \boxed{\mathrm{CMS}_1} \arrow[d, dash, "R_3(\phi_1)"] & \boxed{\mathrm{RS}_1(e^-)} \arrow[l, "B_3(-\eta_e)"'] \arrow[d, dash, "R_3(\phi_1)"] \\
\boxed{\mathrm{RS}(e^+)} \arrow[r, "B_3(\eta_e)"] & \boxed{\mathrm{CMS}} & \boxed{\mathrm{RS}(e^-)} \arrow[l, "B_3(-\eta_e)"']
\end{tikzcd}
$$

(2b)

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