Quantum Field Theory

Lecture Notes

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March 28, 2025

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Chapter 1

Introduction

This lecture has been created to serve as an introduction to Quantum Field Theory for new PhD students in high-energy physics at the IFJ PAN institute. The idea is that it is addressed not only to students interested in theoretical physics but also give those with experimental background.

The first part of this lecture will use scalar field theory, specifically ϕ^4-theory, as a toy theory to develop the basics of Quantum Field Theory. The aim is to develop all necessary tools that are needed to address the most important application of QFT which is the computation of scattering cross sections. ϕ^4-theory is used as the example to avoid various technical complications and to have a clean account. This treatment follows first the cannonical approach introducing perturbation theory and covering the computation of n-point correlation functions. Afterwards the connection with scattering experiments is made through the Lehmann-Symanski-Zimmermann formula. This is largely based on the excellent book by Peskin and Schroeder [1].

Some things that have not yet been covered:

- Pathintegral formalism
- Renormalization
- Higher-order corrections

Chapter 2

Scalar Field Theory

Before we dive in, let's argue why we should consider Quantum Field Theory in the first place and why relativistic quantum mechanics is not enough.

- Relativistic quantum mechanics, a quantised relativistic equation of motion of point particles, leads to negative energy states. In field theory, this issue will naturally resolved by "anti-particles".
- Empirically, we know that particles can be created from energy, most famously stated in $E = mc^2$. Quantum mechanics of point particles can't describe such a process. In field theory, new particles naturally arise in terms of new modes.
- Quantum mechanics features tunnelling (or propagation in equal time), which violates causality. Here, field theory will also provide a natural solution.

Beyond this, there are the following motivations for studying QFT:

- Framework of the most precisely tested fundamental theory of nature.
- The methods used are practical for particle physics and have many applications in solid-state physics, cosmology and statistical physics.

2.1 Classical Field Theory

2.1.1 Principle of least action

We start from classical field theory, which is formulated analogously to classical mechanics in terms of Lagrangians and the principle of least action. The action in classical mechanics is defined by

$$S = \int L\left(x_i(t), \dot{x}_i(t)\right) \mathrm{d}t \,, \tag{2.1}$$

where the Lagrangian $L(x_i(t), \dot{x}_i(t))$ is a scalar function of degrees of freedom x_i and its time derivative $\dot{x}_i$. For now, we assume that no explicit time dependence of L exists and that no higher derivatives appear. The extension of field theory considers *fields* $\phi_i(x)$, which are functions of space-time $x \equiv x^\mu$ as degrees of freedom. For now, we consider scalar fields $\phi(x) : \mathbb{R}^4 \to \mathbb{R}$, but the discussion can be extended to higher-spin fields. Here, we have the action defined as

$$S = \int \mathcal{L}\left(\phi(x), \partial_\mu \phi\right) \mathrm{d}^4 x \,, \tag{2.2}$$

We have the Lagrangian density $\mathcal{L}$ as a straightforward extension of the classical Lagrangian. Like classical mechanics, we impose the *principle of least action*. For this, we define the *variation* of the action δS by parameterized deformations of the fields:

$$S[\phi + \epsilon\delta\phi] - S[\phi] \equiv \epsilon\delta S[\phi, \delta\phi] + \mathcal{O}\left(\epsilon^2\right) . \tag{2.3}$$

Here, the deformation of the field $\delta\phi$ is designed such that they vanish at the boundary of space-time. With this, we have, after expanding in ϵ:

$$\begin{aligned} 0 &= \delta S \\ &= \int \mathrm{d}^4x \left\{ \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) \right\} = \int \mathrm{d}^4x \left\{ \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi - \left(\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\delta\phi + \underbrace{\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right)}_{=\int_{\partial\Omega}\mathrm{d}s_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi(s_\mu)=0} \right\} \\ &= \int \mathrm{d}^4x \left\{ \frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \right\} \delta\phi . \end{aligned} \tag{2.4}$$

This integral has to vanish for *any* $\delta\phi$, which means that the expression in the brackets has to vanish in all points. This gives the Euler-Lagrange equations:

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = 0 \tag{2.5}$$

As an example, which we are going to use throughout the lecture, we will introduce the Lagrangian density for a free scalar field of mass m (which will correspond to a Klein-Gordon field):

$$\mathcal{L}_{\text{K.G.}} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 . \tag{2.6}$$

Plugging this into equation 2.5, we obtain the Klein-Gordon equation:

$$\left(\partial_\mu\partial^\mu + m^2\right)\phi = 0 . \tag{2.7}$$

Solving this equation for a given set of boundary conditions would allow us to predict the field's behaviour ϕ for all time and space coordinates.

2.1.2 Noether's theorem and conservation laws

Like classical mechanics, we can derive conservation laws from continuous transformations of the degrees of freedom under which the Lagrangian density is invariant (up to a total derivative which integrates to zero). Let us consider the following infinitesimal transformation (which is the most general case as all continuous transformations can be expanded in a Taylor series close to unity) parameterized by an infinitesimal parameter α:

$$\phi(x) \to \phi'(x) = \phi(x) + \alpha\Delta\phi , \tag{2.8}$$

$$\mathcal{L}(x) \to \mathcal{L}'(x) = \mathcal{L}(x) + \alpha \underbrace{\partial_\mu J^\mu(x)}_{\equiv\Delta\mathcal{L}} . \tag{2.9}$$

We allow for a term $\Delta\mathcal{L} = \alpha\partial_\mu J^\mu(x)$ because it is a surface term and, as such, leaves the Euler-Lagrange equations invariant. For a given transformation, this term might be 0. This can be compared with the explicit Taylor expansion of the transformation of the Lagrangian under the replacement $\phi \to \phi'$:

$$\mathcal{L}(\phi, \partial_\mu) \to \mathcal{L}(\phi + \alpha\Delta\phi, \partial_\mu(\phi + \alpha\Delta\phi)) \tag{2.10}$$

$$= \mathcal{L}(\phi, \partial_\mu\phi) + \left(\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\alpha}\right)\Bigg|_{\alpha=0} \alpha + \mathcal{O}(\alpha^2) \equiv \mathcal{L}(\phi, \partial_\mu\phi) + \alpha\Delta\mathcal{L} + \mathcal{O}(\alpha^2) \tag{2.11}$$

$$\Delta\mathcal{L} = \left(\frac{\partial\mathcal{L}}{\partial(\phi+\alpha\Delta\phi)}\frac{\partial(\phi+\alpha\Delta\phi)}{\partial\alpha} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu(\phi+\alpha\Delta\phi))}\frac{\partial(\partial_\mu(\phi+\alpha\Delta\phi))}{\partial\alpha}\right)_{\alpha=0} \tag{2.12}$$

$$= \frac{\partial\mathcal{L}}{\partial\phi}\Delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu\Delta\phi = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right) + \underbrace{\left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)}_{=0 \text{ Eq. } 2.5}\Delta\phi \tag{2.13}$$

$$= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right) \tag{2.14}$$

Matching the explicit expression for $\Delta\mathcal{L}$ with equation 2.9, we obtain:

$$\partial_\mu J^\mu = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right) \tag{2.15}$$

$$\Rightarrow \partial_\mu j^\mu(x) = 0 \quad \text{with} \quad j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi - J^\mu \,. \tag{2.16}$$

In other words, we just found a conserved current $j^\mu(x)$ from which we can derive conserved quantities:

$$Q = \int j^0 \mathrm{d}^3x \tag{2.17}$$

An important example is translation invariance. Consider the shift by a constant (but infinitesimal) 4-vector $a^\mu = \alpha e^\mu$ (here α is again an infinitesimal parameter and e^μ an arbitrary vector, these are technically four individual transformations):

$$x^\mu \to x^\mu - \alpha a^\mu \tag{2.18}$$

$$\phi(x) \to \phi(x+a) = \phi + a^\nu\partial_\nu\phi + \mathcal{O}(a^2) \tag{2.19}$$

$$\mathcal{L}(x) \to \mathcal{L}(x) + a^\nu\partial_\nu\mathcal{L} + \mathcal{O}(a^2) = \mathcal{L}(x) + a^\nu\partial_\mu\underbrace{(\delta^\mu_\nu\mathcal{L})}_{\equiv J^\mu_\nu} + \mathcal{O}(a^2) \,. \tag{2.20}$$

Combining all four transformations, we can find the 'stress-energy' tensor:

$$T^\mu_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - \mathcal{L}\delta^\mu_\nu \,. \tag{2.21}$$

A nice by-product of this construction is that it allows additionally to define the Hamiltonian H and the Hamiltonian density $\mathcal{H}$:

$$H = \int T^{00}\mathrm{d}^3x = \int \mathcal{H}\mathrm{d}^3x \tag{2.22}$$

where we have (writing $\partial_0\phi = \dot{\phi}$)

$$\mathcal{H} = T^{00} = \underbrace{\frac{\partial\mathcal{L}}{\partial\dot{\phi}}}_{\equiv\pi}\dot{\phi} - \mathcal{L} = \pi\dot{\phi} - \mathcal{L} \tag{2.23}$$

With the canonical momentum density π.

The Hamiltonian is extremely useful because it allows for a straightforward definition of the quantum mechanics of fields, called *canonical quantisation*. We will first investigate this avenue because it uses many concepts already present in classical mechanics. However, the canonical construction is more complicated (or even impossible) when considering higher-spin fields in the Lagrangian. Therefore, we will also learn a second way to construct a quantum field theory using functional methods.

A second example of using the Noether theorem is rotations to obtain conservation of angular momentum.

2.2 Canonical quantisation

We start with recalling the equation of motions for a system of n particles described by Hamiltonian H in quantum mechanics:

$$[q_i, p_j] = i\delta_{ij} \ , \quad [q_i, q_j] = [p_i, p_j] = 0 \ . \tag{2.24}$$

These equations are sufficient to compute the spectrum of the system given that the Hamiltonian is written in terms of q_i, p_js. The field picture interprets the field's value as the 'canonical field' and the momentum density as the 'canonical momentum'. Therefore, the analogous equations in field theory are:

$$[\phi(\vec{x}), \pi(\vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y}) \quad [\phi(\vec{x}), \phi(\vec{y})] = [\pi(\vec{x}), \pi(\vec{y})] = 0 \ . \tag{2.25}$$

Here $\phi(\vec{x})$ and $\pi(\vec{x})$ are now the field *operators* acting on Hilbert space elements. We start from a Schroedinger picture where the operators are time-independent, and the states are time-dependent. Later on, we will move to the Heisenberg picture, where the operators are time-dependent.

The question now is how to derive the spectrum from these equations. Of course, we need to specify the theory, and the most straightforward option is the free massive scalar field (described by the Klein-Gordon equation).

First, let's start with an observation that we can make if we Fourier decompose the **classical** field $\phi(\vec{x}, t)$[1]:

$$\phi(\vec{x}, t) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \phi(\vec{p}, t) e^{i\vec{p}\vec{x}} \ . \tag{2.26}$$

Since the field is real we have $\phi(\vec{p}, t)^* = \phi(-\vec{p}, t)$. Inserting this in the Klein-Gordon equation yields:

$$(\partial^\mu \partial_\mu + m^2)\phi(\vec{x}, t) = 0 \Rightarrow (\partial_t^2 - \partial^i \partial_i + m^2) \int \frac{\mathrm{d}^3 p}{(2\pi)^3} e^{i\vec{p}\vec{x}} \phi(\vec{p}, t) = 0 \tag{2.27}$$

$$\Rightarrow \int \frac{\mathrm{d}^3 p}{(2\pi)^3} e^{i\vec{p}\vec{x}} (\partial_t^2 + \vec{p}^2 + m^2)\phi(\vec{p}, t) = 0 \tag{2.28}$$

Essentially, we can find the equation of a single harmonic oscillator (SHO) with the frequency of $\omega_{\vec{p}} = \sqrt{|\vec{p}| + m^2}$. The Hamiltonian for a SHO reads:

$$H_{\text{SHO}} = \frac{1}{2} p^2 + \frac{1}{2} \omega^2 q^2 \ . \tag{2.29}$$

We can rewrite this Hamiltonian in terms of creation and annihilation operators:

$$q = \frac{1}{\sqrt{2\omega}}(a + a^\dagger) \ , \quad p = -i\sqrt{\frac{\omega}{2}}(a - a^\dagger) \ , \quad [a, a^\dagger] = 1 \ , \quad H_{\text{SHO}} = \omega(a^\dagger a + \frac{1}{2}) \tag{2.30}$$

[1] This equation also fixes our convention for Fourier transformations in space coordinates $\vec{x}$. For the time coordinate, as we will see later, we will use another minus sign to be complacent with $x \cdot y = x^0 y^0 - x^1 y^1 - x^2 y^2 - x^3 y^3$

In this form, it is trivial to compute the spectrum. So we want something similar for the field *operators*. So, we also write our fields in terms of creation and annihilation operators:

$$\phi(\vec{x}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} (a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}}) \tag{2.31}$$

$$\pi(\vec{x}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3} (-i)\sqrt{\frac{\omega_{\vec{p}}}{2}} (a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} - a^\dagger_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}}) \tag{2.32}$$

The choice of signs in the exponents is needed to ensure that $\phi^\dagger = \phi$, that is, the fields are hermitian operators, which implies that expectation values are physical measurements.

The new operators $a_{\vec{p}}$ and $a^\dagger_{\vec{p}}$ fulfil the following commutation relation:

$$[a_{\vec{p}}, a^\dagger_{\vec{p}'}] = (2\pi)^3 \delta^3(\vec{p} - \vec{p}') \tag{2.33}$$

In the previous section, we constructed the Hamiltonian, which can be written in terms of the canonical variables:

$$H = \int \mathrm{d}^3x (\pi(\vec{x})\dot{\phi}(\vec{x}) - \mathcal{L}) \tag{2.34}$$

$$= \int \mathrm{d}^3x (\pi(\vec{x})\dot{\phi}(\vec{x}) - \frac{1}{2}(\dot{\phi}(\vec{x}))^2 + \frac{1}{2}(\partial_i\phi(\vec{x}))^2 + \frac{1}{2}m^2(\phi(\vec{x}))^2 \tag{2.35}$$

$$= \int \mathrm{d}^3x \frac{1}{2}((\pi(\vec{x}))^2 + (\partial_i\phi(\vec{x}))^2 + m^2(\phi(x))^2) \tag{2.36}$$

We can use the Fourier decompositions of the canonical variables to express the Hamiltonian in terms of the ladder operators (see exercise sheet 1):

$$H = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \omega_{\vec{p}} (a^\dagger_{\vec{p}} a_{\vec{p}} + \frac{1}{2}[a_{\vec{p}}, a^\dagger_{\vec{p}}]) \tag{2.37}$$

This already looks like a harmonic oscillator, so we hope to solve it similarly. The only strange thing is the commuter term $\frac{1}{2}[a_{\vec{p}}, a^\dagger_{\vec{p}}]$, which according to our construction equals ∞. It sums up the zero-point energies at all points in space-time and, therefore, is not quite unexpected. It would shift the spectrum of the theory by an infinite constant, but since measurements can only access the difference between the energy levels of the states, we will never be sensitive to it. Therefore, we will drop it here. Of course, this might make us a bit uneasy, but later on, we will find a systematic way of removing such terms from the theory in terms of "normal ordering".

In any case, we solve the spectrum by investigating the following two commutator

$$[H, a^\dagger_{\vec{p}}] = \omega_{\vec{p}} a^\dagger_{\vec{p}}\,, \quad [H, a_{\vec{p}}] = -\omega_{\vec{p}} a_{\vec{p}} \tag{2.38}$$

These are completely analogous to the SHO case and, therefore, allow the construction of the spectrum in a similar fashion. We start with the ground state $|0\rangle$:

$$\langle 0|0\rangle = 1 \quad \text{with} \quad a_{\vec{p}}\,|0\rangle = 0\,. \tag{2.39}$$

This has an energy of 0 by definition (recall that we have dropped the overall infinite scale). A nontrivial state with energy $\omega_{\vec{p}}$ can be found by acting with $a^\dagger$ on the ground state. We interpret these momentum eigenstates as particles in the momentum representation, they are not localised spatially! We normalise the momentum eigenstates in the following way:

$$|\vec{p}\rangle = \sqrt{2\omega_{\vec{p}}} a^\dagger_{\vec{p}} |0\rangle \tag{2.40}$$

which is then consistent with the interpretation of

$$P^i = -\int \mathrm{d}^3 x \pi(\vec{x}) \partial_i \phi(\vec{x}) (= \int \mathrm{d}^3 x T^{0i}) \tag{2.41}$$

$$= \int \frac{\mathrm{d}p}{(2\pi)^3} \vec{p} a^\dagger_{\vec{p}} a_{\vec{p}} \tag{2.42}$$

$$\tag{2.43}$$

as the total momentum of a state:

$$\vec{P} \left|\vec{p}'\right\rangle = \vec{p}' \left|\vec{p}'\right\rangle \ . \tag{2.44}$$

This motivates us to label the "frequency" $\omega_{\vec{p}}$ as the energy E of a single particle.

We can also write down a completeness relation for 1-particle states:

$$(\mathbf{1})_{\text{1-particle}} = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left|\vec{p}\right\rangle \frac{1}{2E_{\vec{p}}} \left\langle\vec{p}\right| \tag{2.45}$$

Because it will appear in many places, let us give a name to the integral of the form:

$$\mathrm{dLIPS} = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} (2\pi) \delta(p^2 - m^2) \Big|_{p^0 > 0} \tag{2.46}$$

where LIPS stands for Lorentz-Invariant Phase Space.

The discussion so far has used the Schrödinger picture, i.e. we work with time-dependent states and time-independent operators. We can complement this picture by moving to the Heisenberg picture, where the states are time-independent, but the operators are. That's the more natural way because the space and time-dependent field $\phi(x)$ is the central object in the field theory. We can reintroduce the time dependence in the usual way:

$$\phi(x) = \phi(\vec{x}, t) = e^{iHt} \phi(\vec{x}) e^{-iHt} \tag{2.47}$$

which implies:

$$i\partial_t \mathcal{O} = [\mathcal{O}, H] \tag{2.48}$$

$$i\partial_t \phi(x) = i\pi(x) \tag{2.49}$$

$$i\partial_t \pi(x) = -(-\partial_i \partial^i + m^2) \phi(x) \ . \tag{2.50}$$

First of all, we can see that these equations are consistent with the K.-G. Equations:

$$\partial_\mu \partial^\mu + m^2 \phi = 0 \ , \tag{2.51}$$

a good sign. Using our expression of the Schroedinger picture operator $\phi(\vec{x})$ and the fact that $[H, a_{\vec{p}}] = -E_{\vec{p}} a_{\vec{p}}$ we can deduce that:

$$e^{iHT} a_{\vec{p}} e^{-iHt} = a_{\vec{p}} e^{-iE_{\vec{p}} t} \tag{2.52}$$

$$\Rightarrow \phi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}} e^{-ipx} + a^\dagger_{\vec{p}} e^{ipx} \right) \Big|_{p^0 = E_{\vec{p}}} \tag{2.53}$$

2.2.1 Causality

The derived expression for the field operator can now be used to study causality. In quantum mechanics, causality means that measurements of the field (or derived quantities) at two space-like separated points

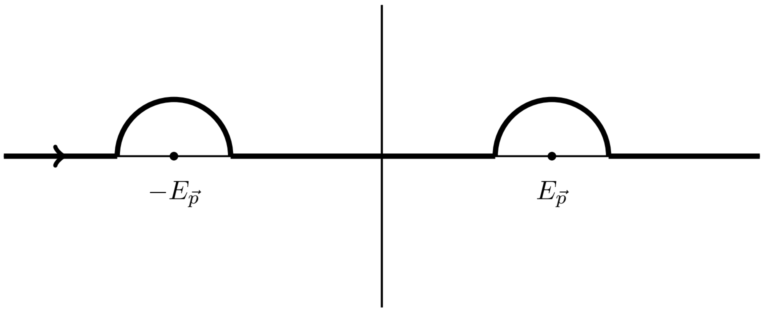

Figure 2.1: Integration contour for the retarded Green's function.

do not influence each other. This can be expressed through the requirement that their commutator vanishes $[\phi(x), \phi(y)] = 0$ if $(x-y)^2 < 0$. Using the expressions for $\phi(x)$ in terms of creation and annihilation operators:

$$\langle 0| [\phi(x), \phi(y)] |0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \frac{1}{\sqrt{2E_{\vec{p}'}}} \langle 0| \left[(a_{\vec{p}} e^{-ipx} + a^\dagger_{\vec{p}} e^{ipx}), (a_{\vec{p}'} e^{-ip'y} + a^\dagger_{\vec{p}'} e^{ip'y}) \right] |0\rangle \tag{2.54}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \frac{1}{\sqrt{2E_{\vec{p}'}}} \langle 0| [a_{\vec{p}}, a^\dagger_{\vec{p}'}] e^{-ipx+ip'y} + [a^\dagger_{\vec{p}}, a_{\vec{p}'}] e^{ipx-ip'y} |0\rangle \tag{2.55}$$

$$= \int \frac{d^3p}{(2\pi)^3} d^3p' \frac{1}{\sqrt{2E_{\vec{p}}}} \frac{1}{\sqrt{2E_{\vec{p}'}}} \langle 0| \delta(\vec{p} - \vec{p}') e^{-ipx+ip'y} - \delta(\vec{p}' - \vec{p}) e^{ipx-ip'y} |0\rangle \tag{2.56}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} (e^{-ip(x-y)} - e^{ip(x-y)}) \tag{2.57}$$

$$= D(x-y) - D(y-x) \tag{2.58}$$

First, we can see that if $(x-y)^2 < 0$, this integral is 0: Each of the two terms is independently Lorentz invariant, and if x, y are space-like, then there is a transformation that $x - y \to -(x-y)$ which makes both terms equal with opposite signs. If both are time-like, then such a transformation is impossible, i.e., the expression will not vanish.

2.2.2 Feynman propagator

For later purposes, we want to introduce the Feynman propagator. Suppose for a moment that $x^0 > y^0$. In that case $x^0 - y^0 > 0$, which implies that $e^{ip^0(x^0-y^0)} \to 0$ for $p^0 \to i\infty$. We can introduce another integration in the complex plane of p^0:

$$\langle 0| [\phi(x), \phi(y)] |0\rangle = \int \frac{d^3p}{(2\pi)^3} \left(\frac{1}{2E_{\vec{p}}} e^{-ip(x-y)}|_{p^0=E_{\vec{p}}} + \frac{1}{-2E_{\vec{p}}} e^{-ip(x-y)}|_{p^0=-E_{\vec{p}}} \right) \tag{2.59}$$

$$= \int \frac{d^3p}{(2\pi)^3} \int \frac{dp^0}{2\pi i} \frac{-1}{p^2 - m^2} e^{-p(x-y)} \; . \tag{2.60}$$

The integration contour used is shown in Figure 2.1.

Further $\langle 0| [\phi(x), \phi(y)] |0\rangle$ can be interpreted as a Green's function. Consider a differential equation given by a linear differential operator L with a source j:

$$Lf(x) = j(x) \tag{2.61}$$

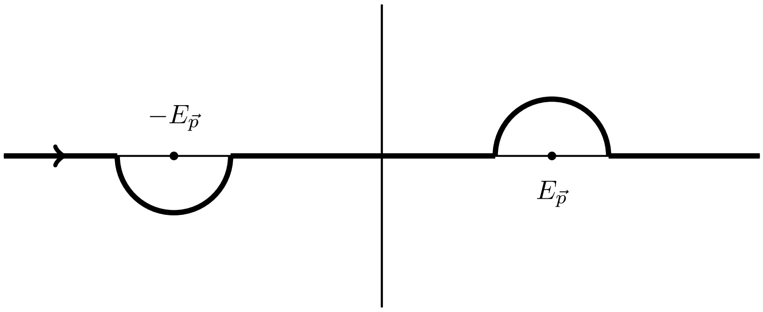

Figure 2.2: Integration contour for the Feynman propagator.

This type of differential equation can be solved formally easily if the function $G(x-y)$ exists such that:

$$LG(x-y) = \delta(x-y) \tag{2.62}$$

If we know G we can construct a solution $f(x)$ as

$$f(x) = \int \mathrm{d}y G(x-y) j(y) \; . \tag{2.63}$$

We want to determine the Green's function for the Klein-Gordon equation:

$$(\partial_\mu \partial^\mu + m^2) G(x-y) = -i\delta^{(4)}(x-y) \tag{2.64}$$

(the $-i$ is just for convention and could be absorbed in $G(x-y)$). We can find it using the Fourier transformation:

$$G(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} e^{-ip(x-y)} \tilde{G}(p) \tag{2.65}$$

$$\text{K.G.} \rightarrow \int \frac{\mathrm{d}^4 p}{(2\pi)^4} (-p^2 + m^2) e^{-ip(x-y)} \tilde{G}(p) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} (-i) e^{-ip(x-y)} \tag{2.66}$$

$$\Rightarrow \tilde{G}(p) = \frac{i}{p^2 - m^2} \tag{2.67}$$

$$\Rightarrow G(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip(x-y)} \tag{2.68}$$

We see that Green's function looks very much like our expression for $\langle 0 | [\phi(x), \phi(y)] | 0 \rangle$, i.e., the Green functions represent propagation amplitudes. There are four different contours that one could evaluate. Each will give rise to other expressions in fields ϕ.

There is one contour of particular interest, which is defined by the Feynman prescription.

$$D_F(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-p(x-y)} \tag{2.69}$$

which is associated with the contour shown in Figure 2.2. Expressed through the 2-point correlator, we can write:

$$D_F(x-y) = \theta(x^0 - y^0) \langle 0 | \phi(x)\phi(y) | 0 \rangle + \theta(y^0 - x^0) \langle 0 | \phi(y)\phi(x) | 0 \rangle \equiv \langle 0 | T\phi(x)\phi(y) | 0 \rangle \tag{2.70}$$

We have introduced the time-ordering symbol T which orders all operators in time such that the latest time is to the left.

2.2.3 Summary

- So far, free field theory! no interactions, just freely propagating fields
- Constructed eigenstates of the Hamiltonian $\rightarrow$ give rise to the interpretation of particles as momentum eigenmodes of the Hamiltonian.
- Boring, nothing is going to happen...

2.3 Interacting fields

Now, we want to move towards interacting field theories. In classical field theory, interactions are introduced within the Lagrangian:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}} \tag{2.71}$$

Linking this to a Hamiltonian containing the interactions is natural in quantum mechanics. Ideally, we would like to have

$$H = H_0 + H_{\text{int}} \tag{2.72}$$

with

$$H_{\text{int}} = \int \mathrm{d}^3x \mathcal{H}_{\text{int}} = -\int \mathrm{d}^3x \mathcal{L}_{\text{int}} \tag{2.73}$$

which however puts some constraints on the structure of $\mathcal{L}_{\text{int.}}$: There is no explicit time and space dependence $\partial\mathcal{L}_{\text{int}}/\partial x^\mu = 0$ for example.

So what would be the general form, then? First, we want to preserve causality, meaning all terms must be 'local', i.e. terms like $\phi(x)\phi(y)$ are not allowed. That would leave us with terms of the form $\phi(x)^n$ or $(\partial\phi)^m$ and any combination thereof. For simplicity, we consider interactions without derivatives for now. That leaves us in the most general case:

$$\mathcal{L}_{\text{int.}} = \sum_n c_n(\phi(x))^n \ . \tag{2.74}$$

We can classify the different terms:

- $n = 0, 1$ trival, just overall shifts of the energy
- $n = 2$ mass term, i.e. c_2 is the mass
- $n = 3, 4$ give rise to ϕ^3 and ϕ^4 theory where c_i are the coupling constatns
- $n \geq 5$ non-renormalisable theories.

What does 'non-renormalable' mean, and why is it a problem? This is far from obvious, and essentially, we draw conclusions that we could only reach much further down the line. In a nutshell, we will see that we cannot solve the interacting theory precisely because of the non-linearities of the equations. Therefore, we will adopt perturbation theory and consider interactions as perturbations of the free theory. However, if we go higher in the perturbative order, we will notice that divergencies are showing up. These divergencies can be treated with renormalisation techniques, which shuffles the divergencies into the coefficients of the

Lagrangian. However, there is a problem if the energy dimension of a coefficient is negative. Essentially, one needs to introduce infinite new interactions to compensate for the divergencies, making such theories less predictive (you have infinite parameters).

To avoid these problems, we will stick to the ϕ^4 theory and can focus on the conceptual development of the treatment of interactions, i.e. we will focus on the ϕ^4-theory Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{2.75}$$

2.3.1 N-point correlation functions

The first field theoretical objects we are going to investigate are (time-ordered) N-point correlations functions, of which the Feynamn-propagator is the $N = 2$ case:

$$\langle\Omega| T\phi(x_1)...\phi(x_n) |\Omega\rangle \tag{2.76}$$

Here, we introduced the vacuum of the interacting theory $|\Omega\rangle$. Understanding that $|\Omega\rangle$ generally differs from the vacuum of the free theory $|0\rangle$ is crucial. The interactions enter in two places: first, in the time dependence of the field,

$$\phi(x) = e^{iHt}\phi(\vec{x})e^{-iHt} \ , \tag{2.77}$$

and second, in the vacuum state $|\Omega\rangle$ itself. How can we tackle the computation? The first hint comes by looking at the equation of motion. We see that a general solution, as we found for the free theory, will not be possible because of the non-linearities in the field, which makes a Fourier analysis unfeasible. We will only ever be able to solve the free theory in general. So, it is intuitive to bring the interactive theory as close as possible to the free theory and treat the interactions as perturbations. A possible way to achieve such a situation, in our ϕ^4 example, is when the interaction strength, parameterized by λ, is small. That will allow us to perform perturbation theory and solve the problem order by order.

The construction starts from the observation that, in the Heisenberg picture, the full Hamiltonian is time-independent. However, this does not imply that its components, i.e. H_0 and $H_{\text{int.}}$ are time-independent as well, but instead, we have:

$$H_0 \equiv H_0(t_) = e^{iHt}H_0(0)e^{-iHt} \quad \text{and} \quad H_{\text{int.}} \equiv H_{\text{int.}}(t) = e^{iHt}H_{\text{int.}}(0)e^{-iHt} \ . \tag{2.78}$$

To work with this, let's define a new set of operators (generally, we consider operators that are polynomials of fields and their derivatives), the *interaction picture* operators, with a unitary transformation of the Heisenberg picture operators as follows:

$$\mathcal{O}_I(t_0) = \mathcal{O}(t_0) \quad \text{and} \quad \mathcal{O}_I(t) = e^{iH_0(t_0)(t-t_0)}\mathcal{O}_I(t_0)e^{-iH_0(t_0)(t-t_0)} \ . \tag{2.79}$$

The time t_0 is arbitrary; we will set it to $t_0 = 0$ for brevity. The reason for this definition is that the unperturbed Hamiltonian, i.e. H_0, is then the same in the Heisenberg picture and interaction picture:

$$H_{0,I}(t) = e^{iH_0(0)t}H_0(0)e^{-iH_0(0)t} = H_0(0) \tag{2.80}$$

and does not depend on the time in the interaction picture (but, of course, in the Heisenberg picture). A second amazing property of this definition is that the dependence of the interaction-picture unperturbed Hamiltonian on the interaction-picture fields is the same as the unperturbed Hamiltonian in the

Heisenberg picture at $t = 0$ (or $t = t_0$):

$$H_0(0) = e^{iH_0(0)t} H_0(0) e^{-iH_0(0)t} = e^{iH_0(0)t} \int \mathrm{d}^3 \mathcal{H}(\phi(0,\vec{x}), \pi(0,\vec{x})) e^{-iH_0(0)t} \tag{2.81}$$

$$= e^{iH_0(0)t} \int \mathrm{d}^3 \mathcal{H}(\phi_I(0,\vec{x}), \pi_I(0,\vec{x})) e^{-iH_0(0)t} \tag{2.82}$$

$$= \int \mathrm{d}^3 \mathcal{H}(\phi_I(t,\vec{x}), \pi_I(t,\vec{x})) \; . \tag{2.83}$$

So, the (field) operators in the interaction picture behave like free fields. We can confirm that by looking at the equation of motion:

$$\frac{\partial}{\partial t} \mathcal{O}_I(t) = \frac{\partial}{\partial t} \left(e^{iH_0(0)t} \mathcal{O}_I(0) e^{-iH_0(0)t} \right) \tag{2.84}$$

$$= iH_0(0)\mathcal{O}_I(t) - i\mathcal{O}_I(t)H_0(0) = -i[\mathcal{O}_I(t), H_0] = -i[\mathcal{O}_I(t), H_{0,I}(t)] \; . \tag{2.85}$$

Indeed, the interaction picture fields are free fields! That means we can use the same ansatz to solve their equation of motion:

$$\phi_I(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}} e^{-ipx} + a^\dagger_{\vec{p}} e^{ipx} \right) \Big|_{x^0=t} \; . \tag{2.86}$$

Finally, we need to relate these interaction picture fields to the actual time-dependent Heisenberg picture fields in our correlator. This can be done formally by the following transformation:

$$\phi(x) = e^{iHt} \phi(0,\vec{x}) e^{-iHt} = e^{iHt} e^{-iH_0(0)t} \phi_I(x) e^{iH_0(0)t} e^{-iHt} \tag{2.87}$$

$$\equiv U^\dagger(t) \phi_I(x) U(t) \; . \tag{2.88}$$

Recovering the dependence on our arbitrary time t_0, we just defined the *time-evolution* operator:

$$U(t,t_0) = e^{iH_0(t_0)(t-t_0)} e^{-iH(t-t_0)} \; . \tag{2.89}$$

Crucially, even though its form suggests it, this operator is not a unitary transformation.

It's all good, but this has not yet helped us evaluate anything. It would be helpful to express the time-evolution operator in terms of $\phi_I(x)$ because everything would be described as 'free' fields, and we are happy. But how?

We will use (as so often) a differential equation to construct this:

$$i\frac{\partial}{\partial t} U(t,t_0) = -(H_0(t_0) e^{iH_0(t_0)(t-t_0)} e^{-iH(t-t_0)} + e^{iH_0(t_0)(t-t_0)} H e^{-iH(t-t_0)}) \tag{2.90}$$

$$= e^{iH_0(t_0)(t-t_0)} (H - H_0(t_0)) e^{-iH(t-t_0)} \tag{2.91}$$

$$= e^{iH_0(t_0)(t-t_0)} H_{\mathrm{int}}(t) e^{-iH(t-t_0)} \tag{2.92}$$

$$= H_{\mathrm{int},I}(t) U(t,t_0) \; . \tag{2.93}$$

Although this differential equation looks simple enough, it is, of course, challenging to solve in general due to the non-linear dependencies in $H_{\mathrm{int},I}(t) = \int \mathrm{d}^3 x \frac{\lambda}{4!} \phi_I^4(x)$. Here, finally, we will use the idea of perturbatively solving this equation with the following ansatz:

$$U(t,t_0) = \mathbb{1} + (-i) \int_{t_0}^{t} \mathrm{d}t_1 H_{\mathrm{int},I}(t_1) + (-i)^2 \int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2 H_{\mathrm{int},I}(t_1) H_{\mathrm{int},I}(t_2) + \dots \tag{2.94}$$

That this solves the equation order by order in λ can be easily verified. To make this solution a bit handier, we will modify this expression. We encounter terms of the form:

$$\int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2 \cdots \int_{t_0}^{t_{n-1}} \mathrm{d}t_n H_{\mathrm{int},I}(t_1) \dots H_{\mathrm{int},I}(t_n) \tag{2.95}$$

which can be massaged using the time order operator T to obtain

$$\cdots = \frac{1}{n!}\int_{t_0}^{t} \mathrm{d}t_1 \ldots \mathrm{d}t_n T\left\{H_{\text{int},I}(t_1)\ldots H_{\text{int},I}(t_n)\right\} \ . \tag{2.96}$$

With this we find a very familiar-looking form for our solution for $U(t,t_0)$:

$$U(t,t_0) = \mathbb{1} + (-i)\int_{t_0}^{t}\mathrm{d}t_1 H_{\text{int},I}(t_1) + \frac{(-i)^2}{2!}\int_{t_0}^{t}\mathrm{d}t_1\mathrm{d}t_2 T\left\{H_{\text{int},I}(t_1)H_{\text{int},I}(t_2)\right\} + \ldots \tag{2.97}$$

$$\equiv T\left\{\exp\left[-i\int_{t_0}^{t}\mathrm{d}t' H_{\text{int},I}(t')\right]\right\} \ . \tag{2.98}$$

It is important to stress that this is a formal way of writing the series expansion. It is not a unitary operator; the series does not converge but is an asymptotic expansion.

Quite naturally, one can generalise the definition of the time-evolution operator to connect two arbitrary times t and t' with $t > t'$:

$$U(t,t') \equiv T\left\{\exp\left[-i\int_{t'}^{t}\mathrm{d}t' H_{\text{int},I}(t')\right]\right\} = e^{iH_0(t_0)(t-t_0)}e^{-iH(t-t')}e^{-iH_0(t_0)(t'-t_0)} \tag{2.99}$$

In exercise sheet 2, we show some of the basic properties of this operator:

$$U(t_1,t_2)U(t_2,t_3) = U(t_1,t_3) \quad \text{and} \quad U(t_1,t_3)\left[U(t_2,t_3)\right]^{\dagger} = U(t_1,t_2) \ . \tag{2.100}$$

With the definition of $U(t,t')$ and $\phi(x)$, we can look again at the correlation function we were interested in. Assume for now that $t_1 > t_2 > \cdots > t_n$ and $t > \max\{t_1, |t_n|\}$:

$$\langle\Omega| T\left\{\phi(t_1)\phi(t_2)\ldots\phi(t_n)\right\}|\Omega\rangle = \langle\Omega|\,\phi(t_1)\phi(t_2)\ldots\phi(t_n)\,|\Omega\rangle \tag{2.101}$$

$$= \langle\Omega|\,U^{\dagger}(t_1,t_0)\phi_I(t_1)U(t_1,t_0)U^{\dagger}(t_2,t_0)\phi_I(t_2)U(t_2,t_0)\ldots U^{\dagger}(t_n,t_0)\phi_I(t_n)U(t_n,t_0)\,|\Omega\rangle \tag{2.102}$$

$$= \langle\Omega|\,U^{\dagger}(t_1,t_0)\phi_I(t_1)U(t_1,t_2)\phi_I(t_2)U(t_2,t_3)\ldots U(t_n,t_{n-1})\phi_I(t_n)U(t_n,t_0)\,|\Omega\rangle \tag{2.103}$$

$$= \langle\Omega|\,U^{\dagger}(t,t_0)U(t,t_1)\phi_I(t_1)U(t_1,t_2)\phi_I(t_2)\ldots U(t_{n-1},t_n)\phi_I(t_n)U(t_n,-t)U(-t,t_0)\,|\Omega\rangle \tag{2.104}$$

$$= \langle\Omega|\,U^{\dagger}(t,t_0)T\{U(t,t_1)\phi_I(t_1)U(t_1,t_2)\phi_I(t_2)\ldots U(t_{n-1},t_n)\phi_I(t_n)U(t_n,-t)\}U(-t,t_0)\,|\Omega\rangle \tag{2.105}$$

$$= \langle\Omega|\,U^{\dagger}(t,t_0)T\{\phi_I(t_1)\phi_I(t_2)\ldots\phi_I(t_n)U(t,-t)\}U(-t,t_0)\,|\Omega\rangle \tag{2.106}$$

The result can't depend on the additional time introduced as a middle step to correct the time ordering.

The last step will be the treatment of $U(-t,t_0)\,|\Omega\rangle$. We can use the following construction to connect with the free vacuum $|0\rangle$. Consider for $\epsilon > 0$ a modified version of the theory:

$$H_\epsilon(t) \equiv H_0(0) + H_{\text{int}}(0)e^{-\epsilon|t|} \ , \quad H_{\epsilon=0}(t) = H_\epsilon(0) = H \ , \quad H_\epsilon(t) \xrightarrow{t\to\pm\infty} H_0(0) \tag{2.107}$$

for which we can study the time dependence of the (Schrodinger picture) state $|\Omega_\epsilon(t)\rangle$:

$$i\frac{\partial}{\partial t}\,|\Omega_\epsilon(t)\rangle = H_\epsilon(t)\,|\Omega_\epsilon(t)\rangle \ , \quad |\Omega_\epsilon(0)\rangle \equiv |\Omega\rangle \tag{2.108}$$

In the limit $t \to \pm\infty$, we can use the adiabatic theorem of Quantum Mechanics, which states that upon a 'slow' change (here controlled by the parameter ϵ), the system stays in a non-degenerate eigenstate of $H_\epsilon(t)$:

$$|\Omega_\epsilon(t)\rangle \xrightarrow{t\to\pm\infty} |0\rangle\,\langle 0|\Omega_\epsilon(t)\rangle + \mathcal{O}(1/t) \tag{2.109}$$

We can multiply this relation by $e^{iH_0(0)t}$ to obtain

$$U_\epsilon(t,0)\left|\Omega\right\rangle \xrightarrow{t\to\pm\infty} \left|0\right\rangle \left\langle 0\right| U_\epsilon(t,0)\left|\Omega\right\rangle \tag{2.110}$$

with the modified time-evolution operator

$$U_\epsilon(t,t_0) = T\exp\left[-i\int_{t_0}^{t}\mathrm{d}t' H_{\mathrm{int}}(t')e^{-\epsilon|t'|}\right]\ , \quad U_{\epsilon=0}(t,t_0) = U(t,t_0) \tag{2.111}$$

Indeed, with this, we can express our correlation in terms of a double limit:

$$\left\langle\Omega\right| T\left\{\phi(t_1)\phi(t_2)\dots\phi(t_n)\right\}\left|\Omega\right\rangle = \left\langle\Omega\right| U^\dagger(t,t_0)T\{\phi_I(t_1)\phi_I(t_2)\dots\phi_I(t_n)U(t,-t)\}U(-t,t_0)\left|\Omega\right\rangle \tag{2.112}$$

$$= \lim_{\epsilon\to 0^+}\lim_{t\to\infty}\left\langle\Omega\right| U_\epsilon^\dagger(t,t_0)T\{\phi_I(t_1)\phi_I(t_2)\dots\phi_I(t_n)U_\epsilon(t,-t)\}U_\epsilon(-t,t_0)\left|\Omega\right\rangle \tag{2.113}$$

$$= \lim_{\epsilon\to 0^+}\lim_{t\to\infty}\left\langle\Omega\right| U_\epsilon^\dagger(t,t_0)\left|0\right\rangle\left\langle 0\right| T\{\phi_I(t_1)\phi_I(t_2)\dots\phi_I(t_n)U_\epsilon(t,-t)\}\left|0\right\rangle\left\langle 0\right| U_\epsilon(-t,t_0)\left|\Omega\right\rangle \tag{2.114}$$

A similar construction works for the case $n=0$, so we find:

$$1 = \left\langle\Omega|\Omega\right\rangle = \lim_{\epsilon\to 0^+}\lim_{t\to\pm 0}\left\langle\Omega\right| U_\epsilon^\dagger(t,t_0)\left|0\right\rangle\left\langle 0\right| U_\epsilon(t,-t)\left|0\right\rangle\left\langle 0\right| U_\epsilon(-t,t_0)\left|\Omega\right\rangle \tag{2.115}$$

Finally, we find the **Gell-Mann-Low** formula by dividing the correlator by $1=\left\langle\Omega|\Omega\right\rangle$:

$$\left\langle\Omega\right| T\left\{\phi(t_1)\phi(t_2)\dots\phi(t_n)\right\}\left|\Omega\right\rangle = \lim_{\epsilon\to 0^+}\lim_{t\to\pm 0}\frac{\left\langle 0\right| T\{\phi_I(t_1)\phi_I(t_2)\dots\phi_I(t_n)U_\epsilon(t,-t)\}\left|0\right\rangle}{\left\langle 0\right| U_\epsilon(t,-t)\left|0\right\rangle} \tag{2.116}$$

$$= \frac{\left\langle 0\right| T\{\phi_I(x_1)\phi_I(x_2)\dots\phi_I(x_n)\exp\left(-i\int\mathrm{d}^4x\mathcal{H}_{\mathrm{int},I}(x)\right)\}\left|0\right\rangle}{\left\langle 0\right| T\{\exp\left(-i\int\mathrm{d}^4x\mathcal{H}_{\mathrm{int},I}(x)\right)\}\left|0\right\rangle}\ . \tag{2.117}$$

Of course, the exponential in this expression is only 'formal'; what we mean is the perturbative expansion in powers of λ, our proxy for a small coupling parameter. This formula can be seen in what we mean by the perturbation theory of QFTs. From now on, we will drop the I to clear the air, implying that all fields and operators we will write are interaction picture fields or operators.

Brilliantly, we now have a formula that we need to expand in a small parameter to approximate the correlation function. There remain, of course, two (if not more) big questions: first, how do I do this in practice? Second, correlation functions are all good, but what can I use them for, or how can I connect them to particle physics? We will first discuss the first question because it will help to set the scene as soon as we discuss scattering processes and the connection between correlation functions and scattering matrix elements.

2.4 Perturbation theory

Unfortunately, we will need to start our career in perturbation theory with some more formalism, and the reason can be seen quite easily when we naively begin to write the expansion of the denominator of the Gell-Mann-Low formula:

$$\left\langle 0\right| T\{\exp\left(-i\int\mathrm{d}^4x\mathcal{H}_{\mathrm{int},I}\right)\}\left|0\right\rangle = 1 + \left\langle 0\right| T\{-i\int\mathrm{d}^4x\mathcal{H}_{\mathrm{int},I}\}\left|0\right\rangle + \dots \tag{2.118}$$

$$= 1 + \left\langle 0\right| T\{-i\int\mathrm{d}^4x\phi^4(x)\}\left|0\right\rangle + \dots \tag{2.119}$$

The second term seems to be proportional to some products of delta functions $[\phi(x),\phi(y)]\sim\delta(x-y)$. Well, that is, of course, not ideal for making any computations. As you will see, these terms will cancel out if one performs the expansion of the ratio rigorously. This is, however, quite cumbersome, and there is a much neater way to see that these infinities cancel out between numerator and denominator through all orders in perturbation theory.

2.4.1 Normal ordering and Wick's theorem

We introduce the notion of normal ordered products of creation and annihilation operators:

$$: a_1^\dagger a_2 a_3^\dagger a_4 := a_1^\dagger a_3^\dagger a_2 a_4 \ . \tag{2.120}$$

An observation we can make for this prescription is:

$$\bar{0} : \cdots : |0\rangle = 0 \ . \tag{2.121}$$

And given the expression for ϕ in terms of creation and annihilation operators:

$$\langle 0| : \phi^n(x) : |0\rangle = 0 \ . \tag{2.122}$$

Great, but what does this buy us? First, let us have another look at our Hamiltonian density for the free theory:

$$H = \int \mathrm{d}^3 x \mathcal{H}(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_{\vec{p}} (a_{\vec{p}}^\dagger a_{\vec{p}} + \frac{1}{2}[a_{\vec{p}}, a_{\vec{p}}^\dagger]) \ . \tag{2.123}$$

If we do the replacement $\mathcal{H} \rightarrow: \mathcal{H}:$ we find:

$$: \mathcal{H} := \int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_{\vec{p}} (: a_{\vec{p}}^\dagger a_{\vec{p}} : + \underbrace{\frac{1}{2} : [a_{\vec{p}}, a_{\vec{p}}^\dagger] :)}_{=0} \tag{2.124}$$

i.e., we find that the problematic $\delta(0)$ term no longer contributes. Similarly, we find for the equation of motion:

$$i \frac{\partial}{\partial} \mathcal{O} = [\mathcal{O}, H] = [\mathcal{O}, : H :] \ , \tag{2.125}$$

since the Hamiltonian and the normal-ordered Hamiltonian only differ by something that commutates with the Hamiltonian. This means, in particular, that the equation of motion does not change when replacing the Hamiltonian with the normal ordered one. That means normal ordered Hamiltonian describe the same physics! Therefore, from now on, we will replace all Hamiltonians with their normal-ordered versions.

That is half the cake already. However, we know that it is crucial to understand the relationship between time ordering and "normal ordering", and this will bring us to Wick's theorem. For concreteness, let's consider the correlator $\langle 0| T\{\phi(x)\phi(y)\} |0\rangle$ (the following argument can, of course, be repeated for more fields) and decompose the fields in a "creation" ϕ^- and "annihilation" ϕ^+ part:

$$\phi(x) = \phi^+(x) + \phi^-(x) \quad \text{implying} \quad \phi^+ |0\rangle = 0 \text{ and } \langle 0| \phi^- = 0 \ . \tag{2.126}$$

Okay, let's have then a look at the time-ordered product of two fields and decompose it. Let's start with the case $x^0 > y^0$

$$T\phi(x)\phi(y) = \langle 0| (\phi^+(x) + \phi^-(x))(\phi^+(y) + \phi^-(y))) \tag{2.127}$$

$$= \underbrace{\phi^+(x)\phi^+(y) + \phi^-(x)\phi^+(y) + \phi^-(x)\phi^-(y) + \quad \phi^-(y)\phi^+(x)}_{\text{normal-ordered}} + [\phi^+(x), \phi^-(y)] \tag{2.128}$$

$$=: \phi(x)\phi(y) : + [\phi^+(x), \phi^-(y)] \tag{2.129}$$

If we would have $y^0 > x^0$ we would end up with the same expression but $[\phi^+(y), \phi^-(y)]$. We will put this together as the definition of the "contraction":

$$\overline{\phi(x)\phi(y)} = \begin{cases} [\phi^+(x), \phi^-(y)] & \text{for } x^0 > y^0 \\ [\phi^+(y), \phi^-(x)] & \text{for } y^0 > x^0 \end{cases} \tag{2.130}$$

This leaves us with:

$$T\phi(x)\phi(y) =: \phi(x)\phi(y) : + \overline{\phi(x)\phi(y)} \tag{2.131}$$

Looking at the expression when taking the vacuum expectation value, we find:

$$iD_F(x-y) = \langle 0| T\phi(x)\phi(y) |0\rangle = \langle 0| \overline{\phi(x)\phi(y)} |0\rangle \tag{2.132}$$

This was now for two fields, but in general, we can find the following (Wick's Theorem):

$$T\{\phi_1 \dots \phi_n\} =: \phi_1 \dots \phi_n : + \sum_{\text{single contr.}} : \phi_1 \dots \phi_n : + \dots + \sum_{\text{fully contr.}} : \phi_1 \dots \phi_n : \tag{2.133}$$

(to be proven in exercise sheet 3, among other identities) some useful identities:

$$: \phi_1 \dots \overline{\phi_k \dots \phi_l} \dots \phi_n : = \overline{\phi_k \phi_l} : \phi_1 \dots \phi_n : |_{\text{without } k,l} \tag{2.134}$$

$$: \phi_1 \dots : \phi_k \dots \phi_l : \dots \phi_n : =: \phi_1 \dots \phi_k \dots \phi_l \dots \phi_n : \tag{2.135}$$

Alrighty, we can have another fresh look at our Gell-Mann-Low formula. For example, a first-order term in ϕ^4 theory:

$$\sim \langle 0| T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) : \phi(z)\phi(z)\phi(z)\phi(z) :\} |0\rangle \tag{2.136}$$

$$= \underbrace{\langle 0| : \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\phi(z)\phi(z)\phi(z)\phi(z) : |0\rangle}_{=0} + \dots + \tag{2.137}$$

$$+ \langle 0| \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\phi(z)\phi(z)\phi(z)\phi(z) |0\rangle \tag{2.138}$$

Of course, all terms with a normal-ordered factor are zero; thus, only the fully contracted terms survive! Hence, we can express this in terms of propagators! Still, some contributions seem a bit odd because they are disconnected. There are two sets of disconnected diagrams, first there are diagrams where external are not all connected with each other but all fields from the interaction Hamiltonian are connected to the external fields. These terms are non vanishing and we will encounter them later on when we will talk about scattering theory. The second type of disconnected diagrams are contributions where fields from the interaction Hamiltonian are not connected to any external field. We will refer to these as vacuum contributions. These do not contribute to the correlator, as we show in the next section.

2.4.2 Vacuum contributions

We can deal with them and the numerator of the Gell-Mann-Low formula in one go. For that we come back to our expression for the correlator, starting from k's term in the numerator of the Gell-Mann-Low formula:

$$\frac{(-i)^k}{k!} \int \mathrm{d}z_1 \dots \mathrm{d}z_k T\left[\phi_1 \dots \phi_n \mathcal{H}_{\text{int}}(z_1) \dots \mathcal{H}_{\text{int}}(z_k)\right] = \sum \text{all possible full contractions} . \tag{2.139}$$

Among all these contractions there are those that just will have contractions among different $\mathcal{H}_{\text{int}}(z_i)$ (these are called the vaccuum componenets). Assume now we have k_2 ($k_1 = k - k_2$ of these Hamiltonians that are only connected among each other, while the rest is contracted with external fields in some way. There are, as discussed above,

$$\begin{pmatrix} k \\ k_2 \end{pmatrix} = \frac{k!}{k_1! k_2!} \tag{2.140}$$

such terms. Summing these up, we can, after relabelling of the unrelated z_i's, split up those terms in the following way:

$$\cdots \ni \frac{(-i)^{k_1}}{k_1!} \int \mathrm{d}z_1 \ldots \mathrm{d}z_{k_1} \phi_1 \ldots \phi_n \mathcal{H}_{\text{int}}(z_1) \ldots \mathcal{H}_{\text{int}}(z_{k_1}) \cdot \tag{2.141}$$

$$\frac{(-i)^{k_2}}{k_2!} \int \mathrm{d}z_1 \ldots \mathrm{d}z_{k_2} \mathcal{H}_{\text{int}}(z_1) \ldots \mathcal{H}_{\text{int}}(z_{k_2}) \tag{2.142}$$

Of course for each power k_1 we find the infinite series in k_2. Thus we can write:

$$\langle 0| T\{\phi_I(x_1)\phi_I(x_2)\ldots\phi_I(x_n) \exp\left(-i \int \mathrm{d}^4x \mathcal{H}_{\text{int},I}(x)\right)\} |0\rangle = \tag{2.143}$$

$$\langle 0| T\{\phi_I(x_1)\phi_I(x_2)\ldots\phi_I(x_n) \exp\left(-i \int \mathrm{d}^4x \mathcal{H}_{\text{int},I}(x)\right)\} |0\rangle \Big|_{\text{no vacuum components}} \tag{2.144}$$

$$\cdot \langle 0| T\{\exp\left(-i \int \mathrm{d}^4x \mathcal{H}_{\text{int},I}(x)\right)\} |0\rangle \tag{2.145}$$

which exactly factors out the numerator of the Gell-Mann-Low formula. This means we are left with the following version:

$$\langle \Omega| T\{\phi(t_1)\phi(t_2)\ldots\phi(t_n)\} |\Omega\rangle = \tag{2.146}$$

$$\langle 0| T\{\phi(x_1)\phi(x_2)\ldots\phi(x_n) \exp\left(-i \int \mathrm{d}^4x \mathcal{H}_{\text{int}}(x)\right)\} |0\rangle \Big|_{\text{no vacuum components}} . \tag{2.147}$$

This formula can now be used to do actual computations. We will see that this expression allow for an efficient and intuitive graph interpretations and will use that often to perform actual compuations.

2.4.3 Feynman diagrams

An efficient way to write down non-vanishing contributions to correlator is given by writing down Feynman diagrams. This method is based essentially on the relation between the contraction and the Feynman propagator:

$$iD_F(x-y) = \langle 0| T\phi(x)\phi(y) |0\rangle = \langle 0| \overline{\phi(x)\phi}(y) |0\rangle \ . \tag{2.148}$$

Let's consider as an example the order λ term for the $n = 4$ case:

$$\frac{-i\lambda}{4!} \langle 0| \int \mathrm{d}z T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) : \phi(z)\phi(z)\phi(z)\phi(z) :\} |0\rangle \tag{2.149}$$

$$= \frac{-i\lambda}{4!} \int \mathrm{d}z 4! D_F(x_1 - z) D_F(x_2 - z) D_F(x_3 - z) D_F(x_4 - z) \ . \tag{2.150}$$

The additional factor 4! arises from all combinations to connect the fields $\phi(z)$ to the external fields. This immediately allows us to relate this to a graph where 4 external edges (x_i) are connected to a vertex at position z. The vertex is associated with a factor $-i\lambda$.

One can imagine how this plays out for more complicated correlators and higher loop terms. We will define the following set of Feynman rules for ϕ^4 theory in position space:

1. external fields come with a propagator iD_F.

2. each vertex contributes with a factor $i\lambda$.

3. integrate over the position of all vertices.

Symmetry factors

For this, we will first try to imagine what kind of contributions there could be. Consider two different Hamiltonians $\mathcal{H}_{1,\text{int}}$ and $\mathcal{H}_{2,\text{int}}$ each contributing a term of the type

$$\frac{1}{k!}\phi_1^k(z_1) \quad \text{and} \quad \frac{1}{l!}\phi_2^l(z_2)\ . \tag{2.151}$$

There are

$$\frac{1}{k!}\frac{1}{n!}\underbrace{\begin{pmatrix} k \\ n \end{pmatrix}}_{\text{Choose n out of k}}\begin{pmatrix} l \\ n \end{pmatrix} n! = \frac{1}{(k-n)!}\frac{1}{(l-n)!}\frac{1}{n!} \tag{2.152}$$

combinations possible. Similarly, for contractions within one single term, we have the following.

$$\frac{1}{k!}\underbrace{\begin{pmatrix} k \\ 2n \end{pmatrix}}_{\text{Choose 2n out of k}}\begin{pmatrix} 2n \\ 2 \end{pmatrix}\begin{pmatrix} 2n-2 \\ 2 \end{pmatrix}\dots\begin{pmatrix} 2 \\ 2 \end{pmatrix} n! = \frac{1}{(k-2n)!}\frac{1}{2^n n!} \tag{2.153}$$

combinations.

Feynman rules in momentum space

Eventually, when are computing scattering processes, we want to perform computations in momentum space. We can convert the correlator in position space to momentum space via Fourier transformation:

$$\int \prod_{i=1}^{n} \mathrm{d}x_i e^{i\sum_{j=1}^{n} p_j x_j} \langle\Omega| T\{\phi(x_1)\dots\phi(x_n)\}|\Omega\rangle \tag{2.154}$$

$$= \int \prod_{i=1}^{n} \mathrm{d}x_i e^{i\sum_{j=1}^{n} p_j x_j} \langle\Omega| T\{\phi(x_1-x_n)\dots\phi(0)\}|\Omega\rangle \tag{2.155}$$

$$= \int \prod_{i=1}^{n} \mathrm{d}x_i e^{i\sum_{j=1}^{n-1} p_j x_j} e^{-i\sum_{j=1}^{n} p_j x_n} \langle\Omega| T\{\phi(x_1-x_n)\dots\phi(0)\}|\Omega\rangle \tag{2.156}$$

$$= (4\pi)^4\delta^{(4)}(\sum_{k=0}^{n} p_k) \underbrace{\int \prod_{i=1}^{n-1} \mathrm{d}x_i e^{i\sum_{j=1}^{n-1} p_j x_j} \langle\Omega| T\{\phi(x_1-x_n)\dots\phi(0)\}|\Omega\rangle}_{\equiv G_{\phi_1,\phi_2,\dots}(p_1,\dots,p_{n-1})} \tag{2.157}$$

The inverse is given by

$$\langle\Omega| T\{\phi(x_1)\dots\phi(x_n)\}|\Omega\rangle = \int \prod_{i=1}^{n-1} \frac{\mathrm{d}^4 p_i}{(2\pi)^4} e^{-i\sum_{j=1}^{n-1} p_j(x_j-x_n)} G_{\phi\dots}(p_1,\dots,p_{n-1})\ . \tag{2.158}$$

The propagtor in momentum space can easily read from the definition:

$$\langle 0| T\{\phi(x)\phi(y)\}|0\rangle \equiv \int \frac{\mathrm{d}^4 p}{(2\pi)^4} e^{-i(x-y)} G(p) \Rightarrow iG(p) = \frac{i}{p^2-m^2+i0^+} \tag{2.159}$$

Literature

Recommended monographs

[1] Michael E. Peskin and Daniel V. Schroeder. *An Introduction to quantum field theory*. Reading, USA: Addison-Wesley, 1995. ISBN: 978-0-201-50397-5.