Precision phenomenology with multi-jet final states at the LHC

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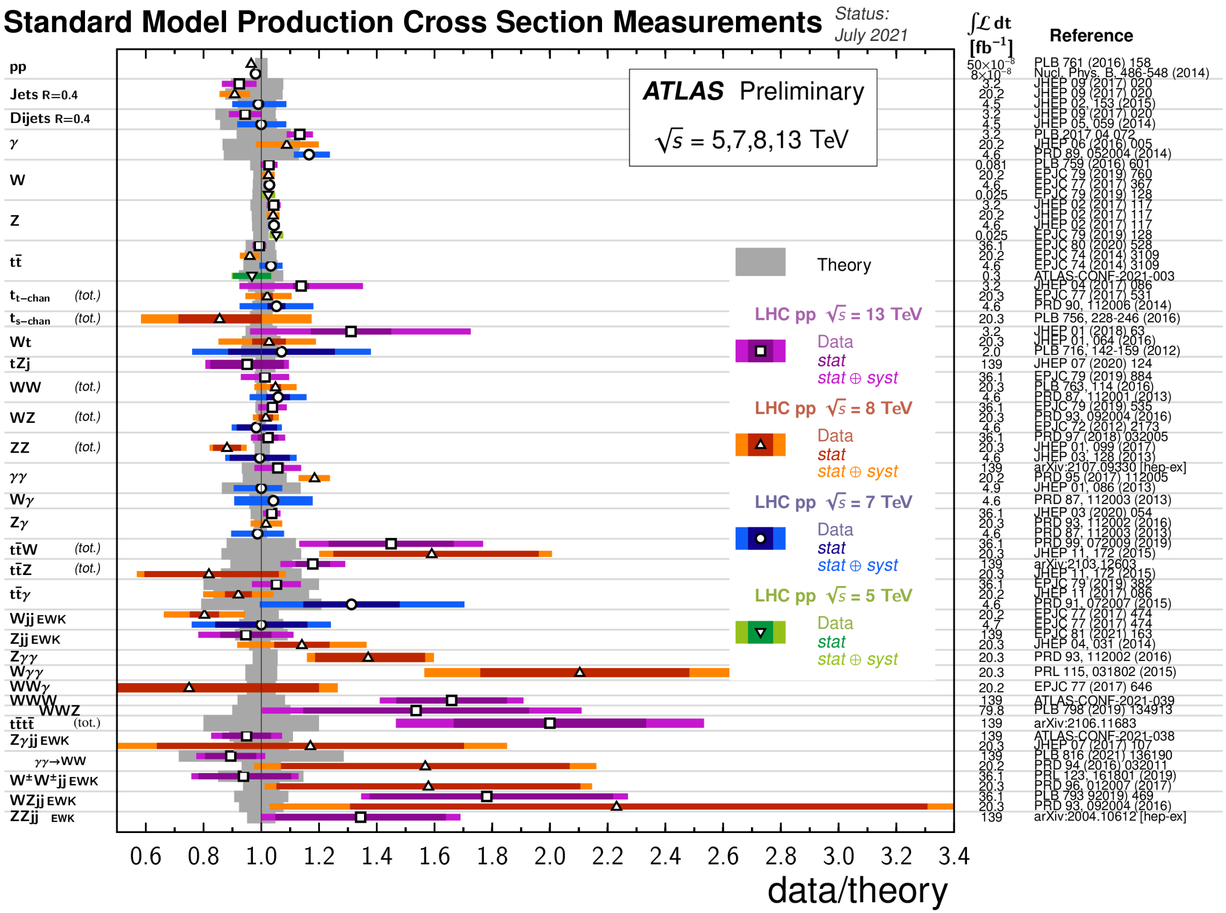
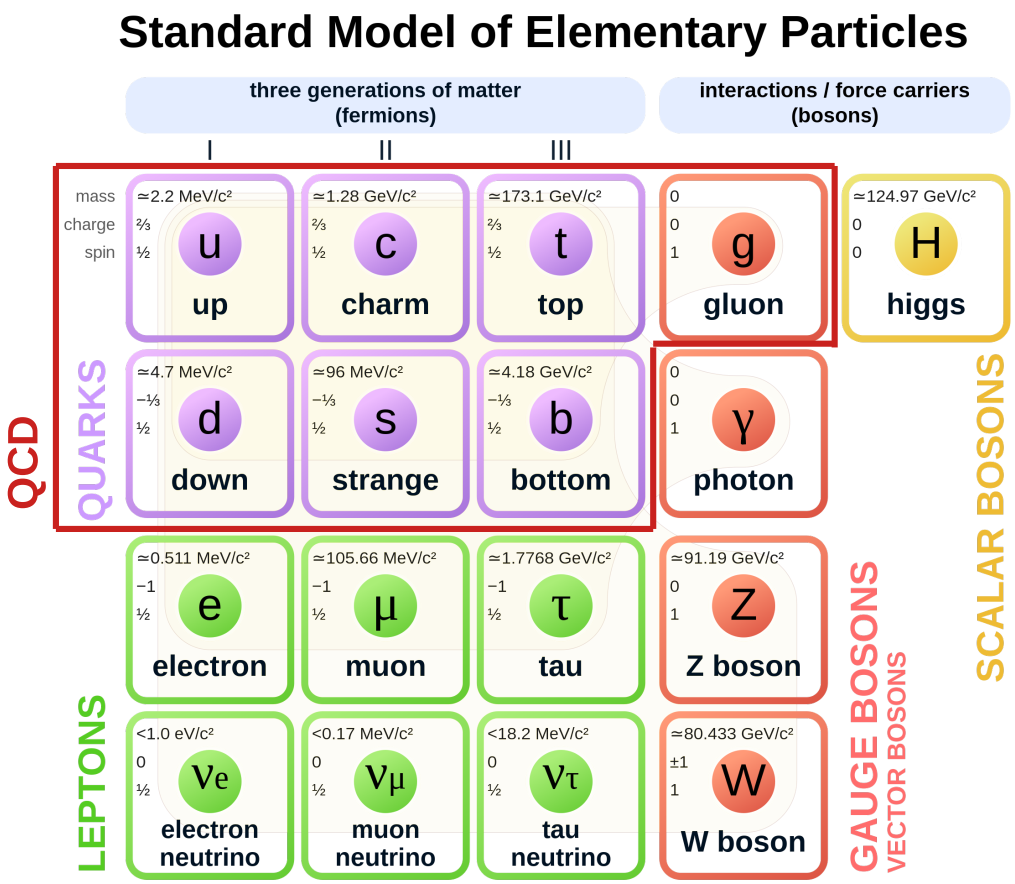
Topics

- Multi-jet observables/event shapes at hadron colliders
- Extraction of the strong coupling constant
- Wider context of my research

Precision era of the LHC



Precision era of the LHC



- Collider data constrains the various interactions in the Standard Model.
- At the LHC QCD is part of any process!
  1) The limiting factor in many analyses is QCD and associated uncertainties.
     → Radiative corrections indispensable
  2) How well we do know QCD? Coupling constant, running, PDFs, ...
- The production of high energy jets allow to probe pQCD at high energies directly

$$\mathcal{L}_{\mathrm{QCD}} = \bar{q}_i(\gamma^\mu \mathcal{D}_\mu - m_i)q_i - \frac{1}{4} F_a^{\mu\nu} F_{\mu\nu}^a$$

1) Testing the predicted dynamics
2) Extract the coupling constant

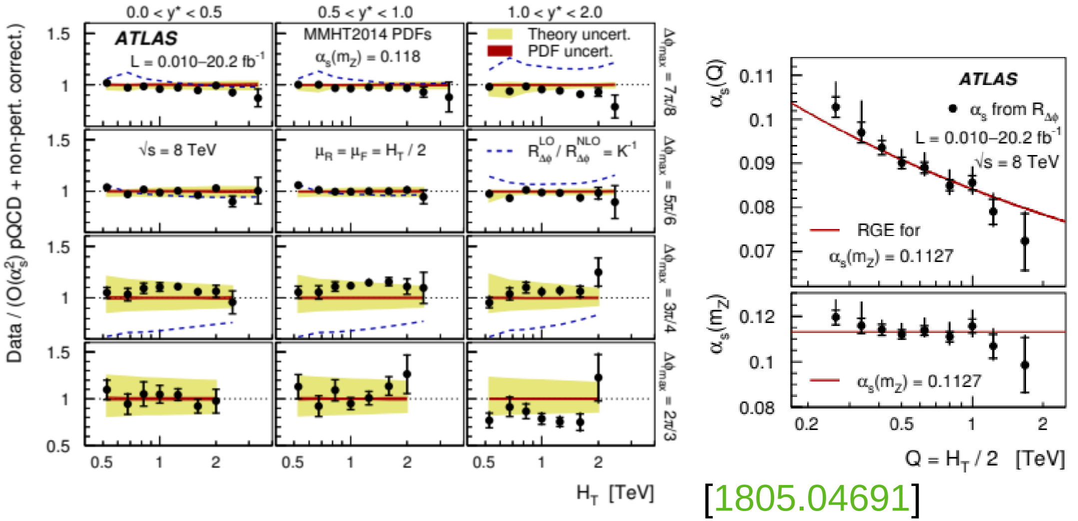
Phenomenology with jet observables

Multi-jet: R32 ratios, event-shapes
Tests of pQCD, α_S extraction

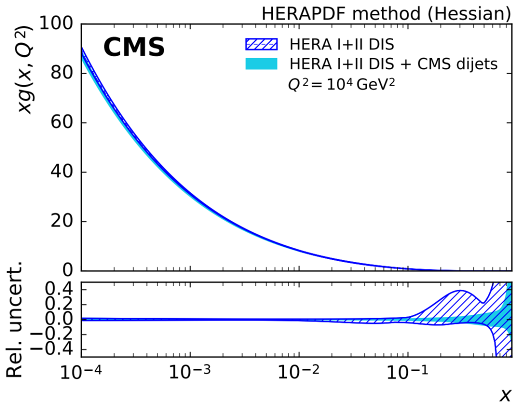
Single inclusive/two-jet
PDF +α_S extraction

Direct BSM

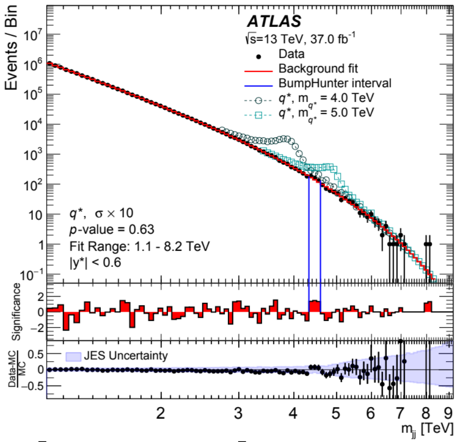
Three-jet production leading contribution



[1805.04691]



[1705.02628]



[1703.09127]

Precision theory required! → pQCD

Data driven

Multi-jet observables

Uncertainties in theory large compared to experiment

- NNLO QCD needed for precise theory-data comparisons

  → Restricted precision QCD studies to two-jet data

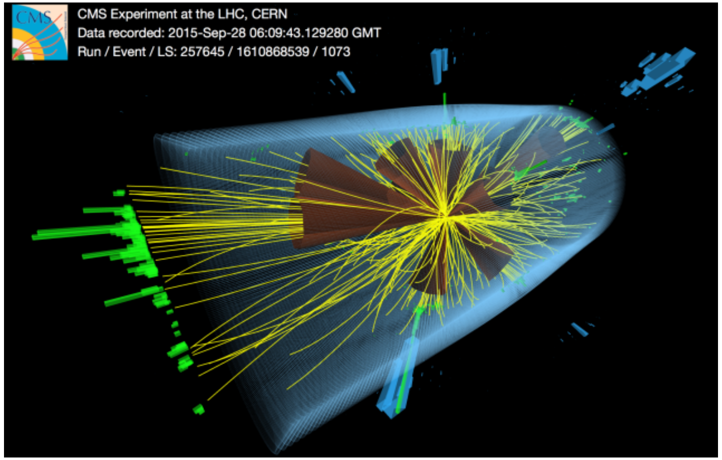
- New NNLO QCD three-jet computations give access to many more observables:
  - Jet ratios, for example R32:

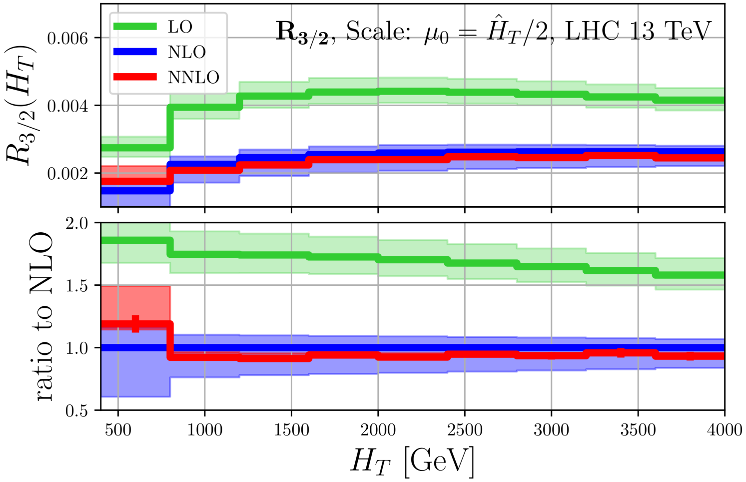
Next-to-Next-to-Leading Order Study of Three-Jet Production at the LHC
Czakon, Mitov, Poncelet [2106.05331]

$$R^i(\mu_R, \mu_F, \mathrm{PDF}, \alpha_{S,0}) = \frac{\mathrm{d}\sigma_3^i(\mu_R, \mu_F, \mathrm{PDF}, \alpha_{S,0})}{\mathrm{d}\sigma_2^i(\mu_R, \mu_F, \mathrm{PDF}, \alpha_{S,0})}$$

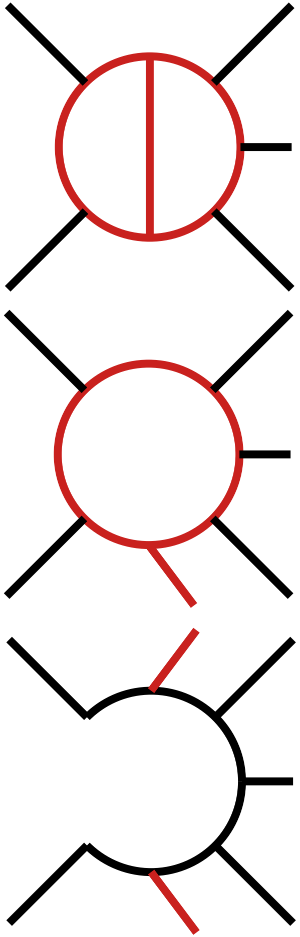
  - Event shapes (based on particles or jets)

NNLO QCD corrections to event shapes at the LHC
Alvarez, Cantero, Czakon, Llorente, Mitov, Poncelet 2301.01086





NNLO QCD prediction beyond 2 → 2



2 → 3 Two-loop amplitudes

- (Non-) planar 5 point massless [Chawdry'19'20'21,Abreu'20'21,Agarwal'21,Badger'21]
  → triggered by efficient MI representation [Chicherin'20]
- For three-jets → [Abreu'20'21] (checked against NJET [Badger'12'21])
- 5 point with one external mass [Abreu'20,Syrrakos'20,Canko'20,Badger'21'22,Chicherin'22]

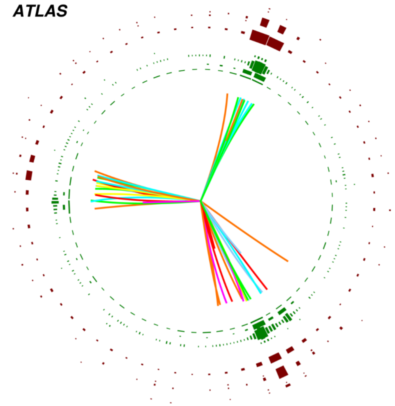
One-loop amplitudes → OpenLoops [Buccioni'19]

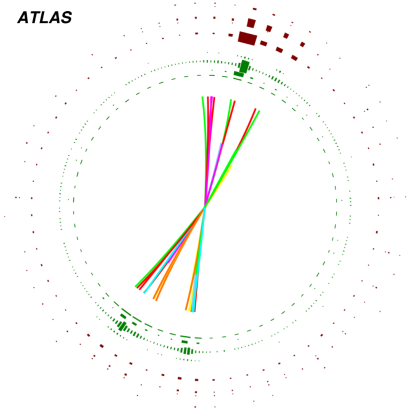
- Many legs and IR stable (soft and collinear limits)

Double-real Born amplitudes → AvHlib[Bury'15]

- IR finite cross-sections → NNLO subtraction schemes
  qT-slicing [Catani'07], N-jettiness slicing [Gaunt'15/Boughezal'15], Antenna [Gehrmann'05-'08], Colorful [DelDuca'05-'15], Projetction [Cacciari'15], Geometric [Herzog'18], Unsubtraction [Aguilera-Verdugo'19], Nested collinear [Caola'17], Local Analytic [Magnea'18], **Sector-improved residue subtraction** [Czakon'10-'14,'19]

Encoding QCD dynamics in event shapes





Using (global) event information to separate different regimes of QCD event evolution:

- **Thrust & Thrust-Minor**

$$T_\perp = \frac{\sum_i |\vec{p}_{T,i} \cdot \hat{n}_\perp|}{\sum_i |\vec{p}_{T,i}|}\,, \quad \text{and} \quad T_m = \frac{\sum_i |\vec{p}_{T,i} \times \hat{n}_\perp|}{\sum_i |\vec{p}_{T,i}|}\,.$$

- (Transverse) Linearised Sphericity Tensor

$$\mathcal{M}_{xyz} = \frac{1}{\sum_i |\vec{p}_i|} \sum_i \frac{1}{|\vec{p}_i|} \begin{pmatrix} p_{x,i}^2 & p_{x,i}p_{y,i} & p_{x,i}p_{z,i} \\ p_{y,i}p_{x,i} & p_{y,i}^2 & p_{y,i}p_{z,i} \\ p_{z,i}p_{x,i} & p_{z,i}p_{y,i} & p_{z,i}^2 \end{pmatrix}$$

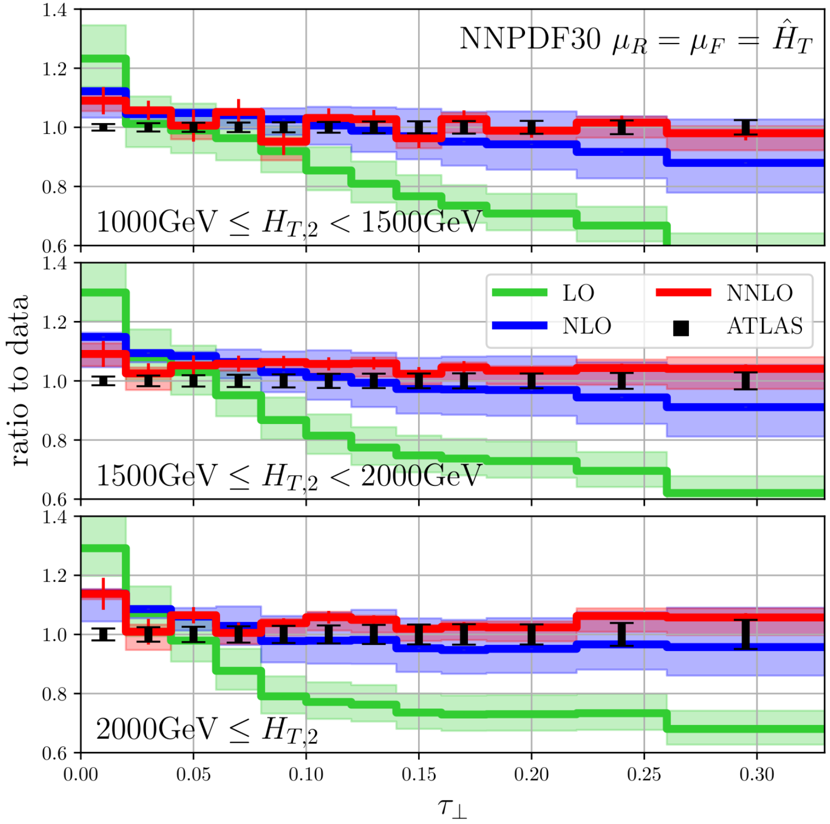
- **Energy-energy correlators**
- N-Jettiness
- Generalised event shapes → Earth-Mover Distance

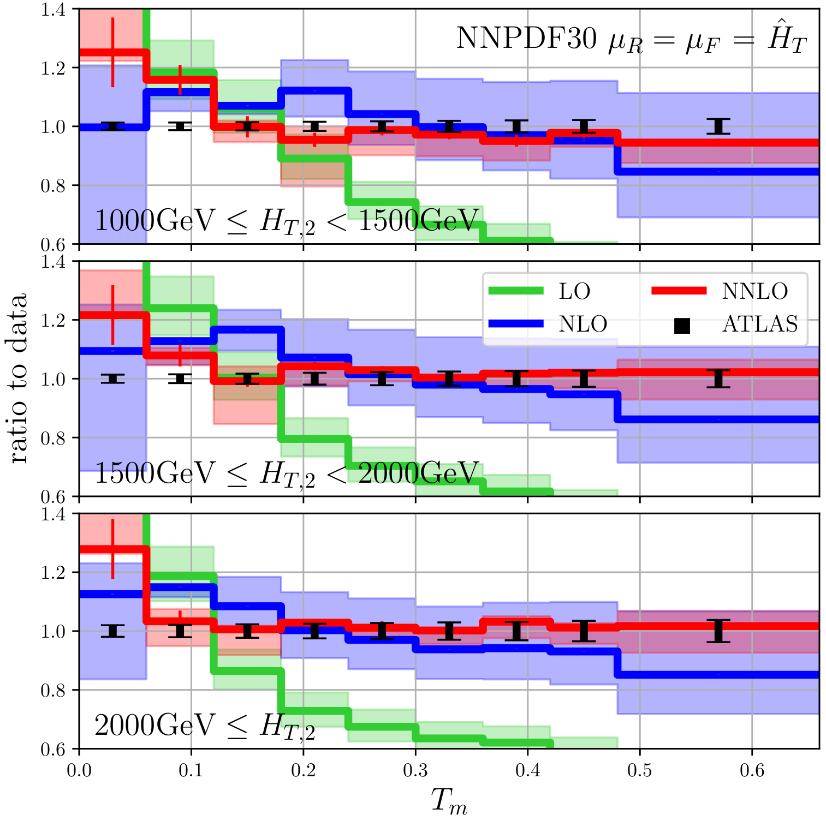
Here: use jets as input → experimentally advantageous (better calibrated, smaller non-pert.)

Transverse Thrust @ NNLO QCD

NNLO QCD corrections to event shapes at the LHC
Alvarez, Cantero, Czakon, Llorente, Mitov, Poncelet 2301.01086

ATLAS [2007.12600]

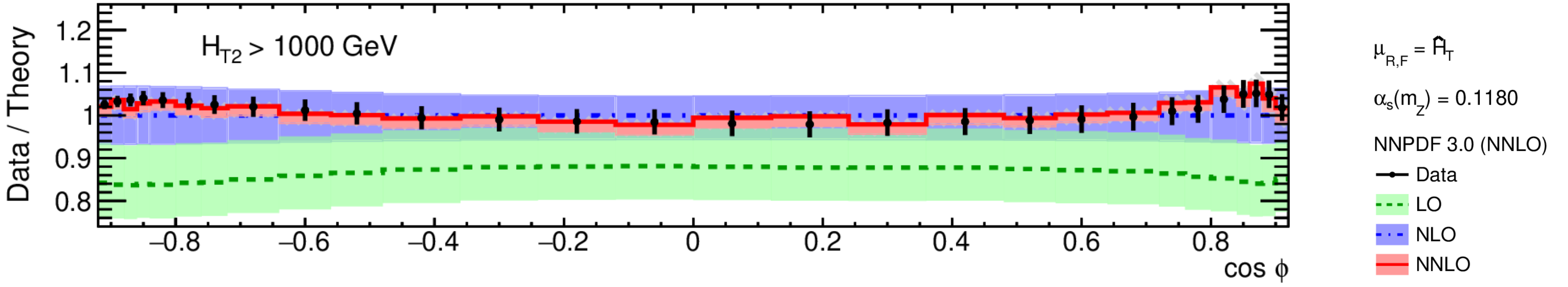




The transverse energy-energy correlator

$$\frac{1}{\sigma_2}\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\Delta\phi} = \frac{1}{\sigma_2}\sum_{ij}\int \frac{\mathrm{d}\sigma\; x_{\perp,i}x_{\perp,j}}{\mathrm{d}x_{\perp,i}\mathrm{d}x_{\perp,j}\mathrm{d}\cos\Delta\phi_{ij}}\delta(\cos\Delta\phi - \cos\Delta\phi_{ij})\mathrm{d}x_{\perp,i}\mathrm{d}x_{\perp,j}\mathrm{d}\cos\Delta\phi_{ij}\,,$$

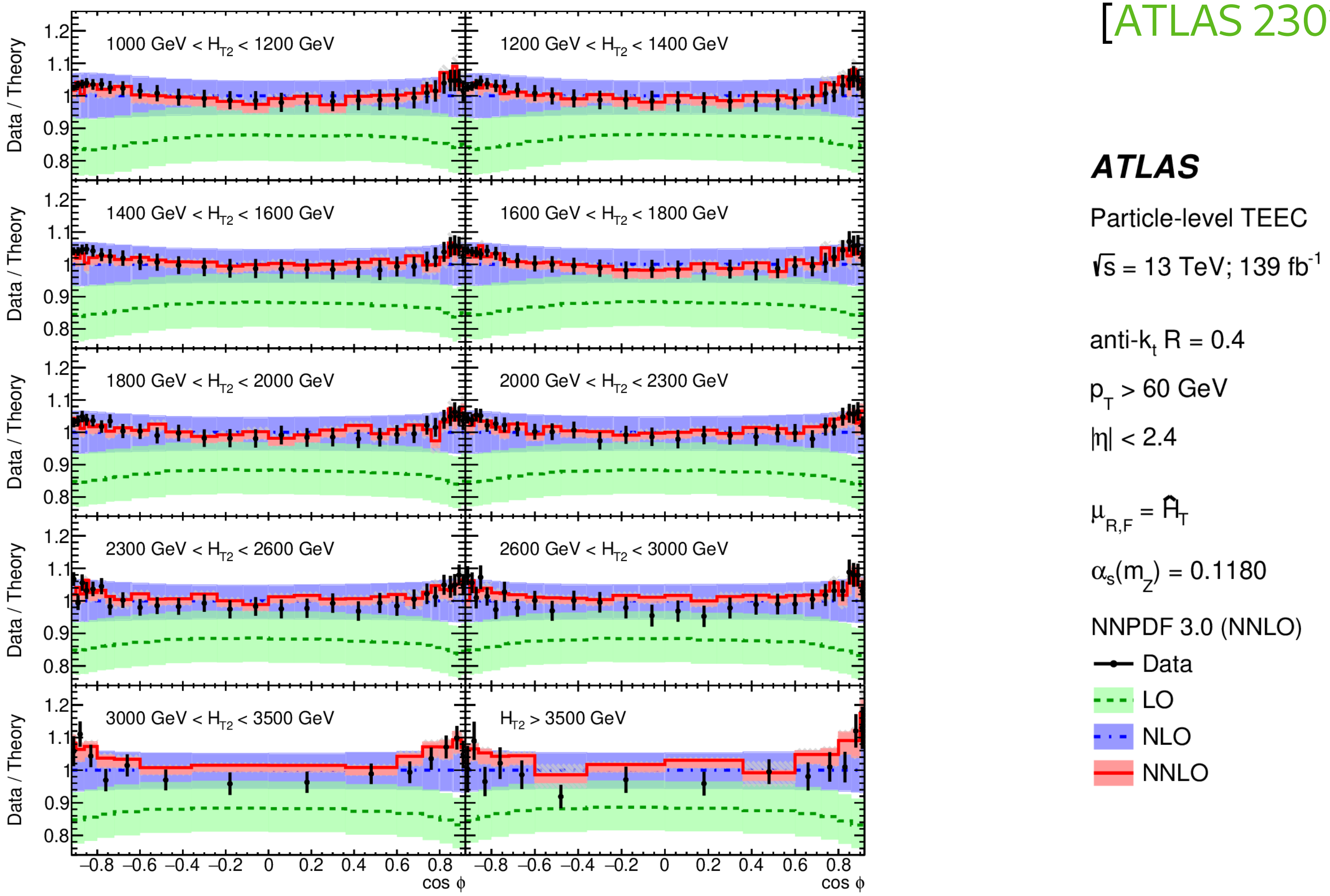
- Insensitive to soft radiation through energy weighting
- Event topology separation:
  - Central plateau contain isotropic events
  - To the right: self-correlations, collinear and in-plane splitting
  - To the left: back-to-back



[ATLAS 2301.09351]

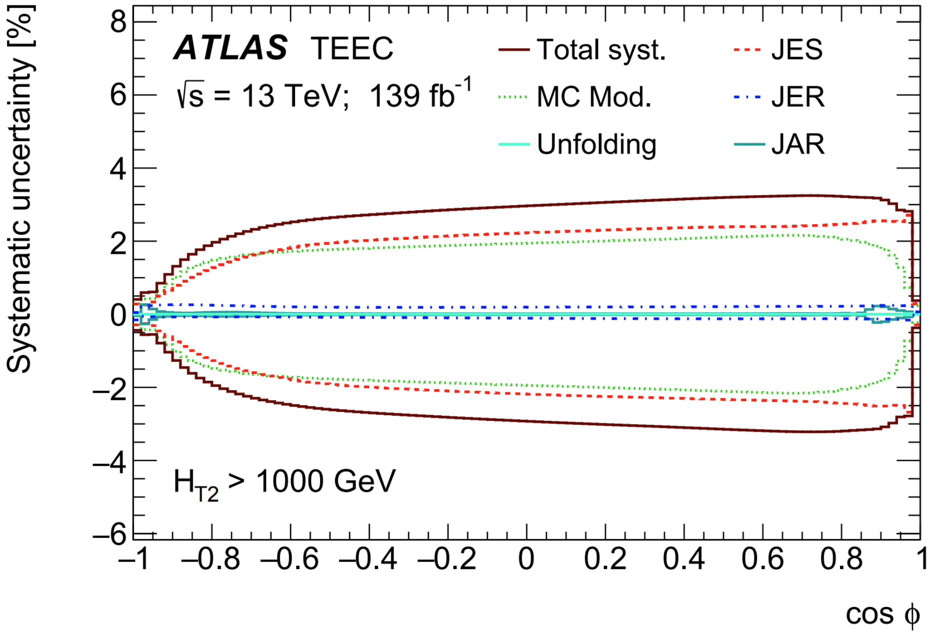
Double differential TEEC

[ATLAS 2301.09351]

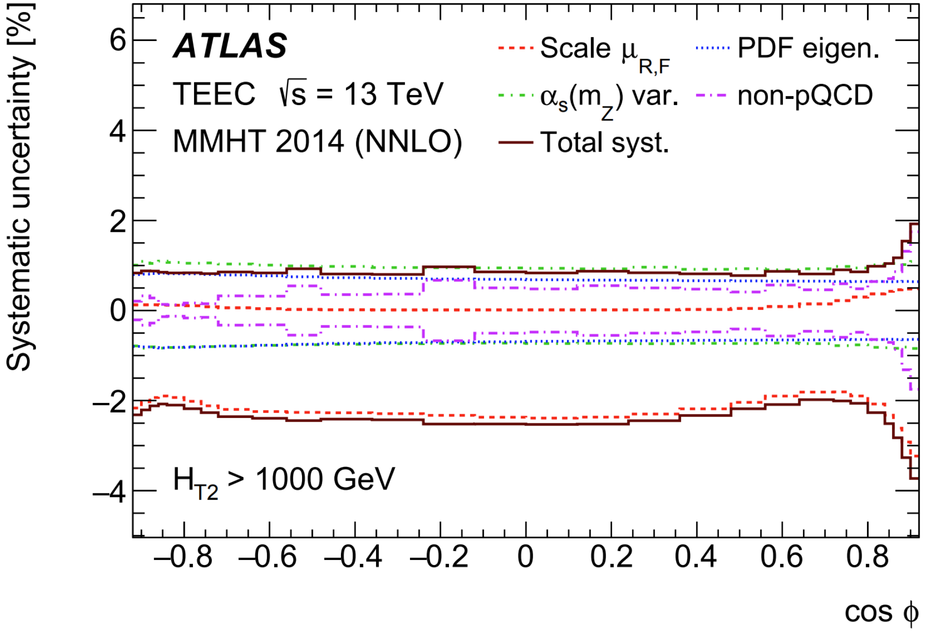


Systematic Uncertainties TEEC

Experimental uncertainties



Theory uncertainties



Scale dependence is the dominating uncertainty → NNLO QCD required to match exp.

Sensitivity to the strong coupling constant

- R32 ratio: $R^i(\mu_R, \mu_F, \mathrm{PDF}, \alpha_{S,0}) = \frac{\mathrm{d}\sigma_3^i(\mu_R, \mu_F, \mathrm{PDF}, \alpha_{S,0})}{\mathrm{d}\sigma_2^i(\mu_R, \mu_F, \mathrm{PDF}, \alpha_{S,0})}$
- Using the strong coupling's running: $\alpha_S(\mu_R, \alpha_{S,0}) = \alpha_{S,0}\left(1 - \alpha_{S,0} b_0 \ln\left(\frac{\mu_R^2}{m_Z^2}\right) + \mathcal{O}({\alpha_{S,0}}^2)\right)$
- Absorb running in the perturbative expansion → linear dependence

$$R^{\mathrm{NNLO}}(\mu, \alpha_{S,0}) = \frac{\mathrm{d}\sigma_3^{\mathrm{NNLO}}(\mu, \alpha_{S,0})}{\mathrm{d}\sigma_2^{\mathrm{NNLO}}(\mu, \alpha_{S,0})}$$

$$= \frac{{\alpha_{S,0}}^3\left(\mathrm{d}\tilde{\sigma}_3^{(0)}(\mu) + \alpha_{S,0}\mathrm{d}\tilde{\sigma}_3^{(1)}(\mu) + {\alpha_{S,0}}^2\mathrm{d}\tilde{\sigma}_3^{(2)}(\mu) + \mathcal{O}({\alpha_{S,0}}^3)\right)}{{\alpha_{S,0}}^2\left(\mathrm{d}\tilde{\sigma}_2^{(0)}(\mu) + \alpha_{S,0}\mathrm{d}\tilde{\sigma}_2^{(1)}(\mu) + {\alpha_{S,0}}^2\mathrm{d}\tilde{\sigma}_2^{(2)}(\mu) + \mathcal{O}({\alpha_{S,0}}^3)\right)}\,.$$

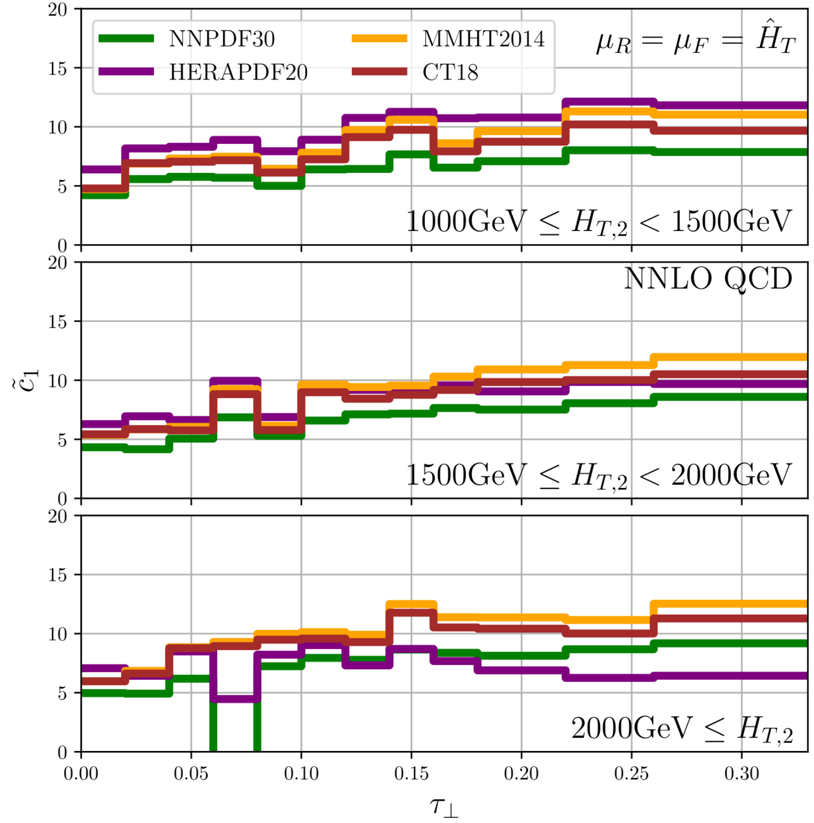
- In practice using LHAPDF running and perform fit to Taylor expansion around $\alpha_s = 0.118$:

$$R^{\mathrm{NNLO,fit}}(\mu, \alpha_{S,0}) = c_0 + c_1(\alpha_{S,0} - 0.118) + c_2(\alpha_{S,0} - 0.118)^2 + c_3(\alpha_{S,0} - 0.118)^3$$

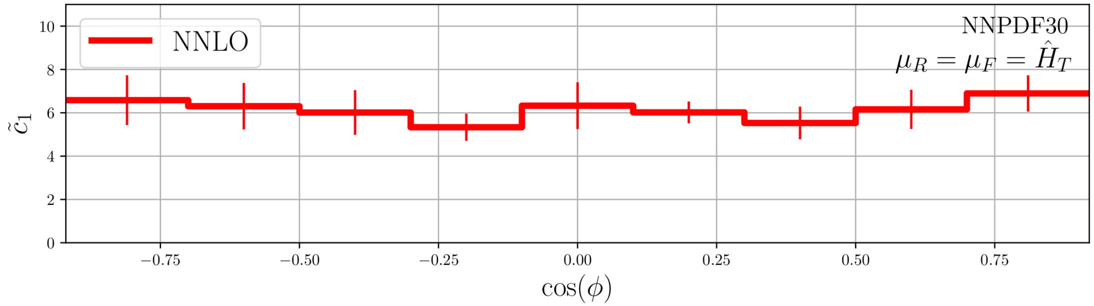
mostly linear dependence

Strong coupling dependence

Thrust



TEEC



Visualisation of α_S dependence

$$\tilde{c}_1 = \frac{c_1}{R^{\text{NNLO}}(\alpha_{S,0} = 0.118)}$$

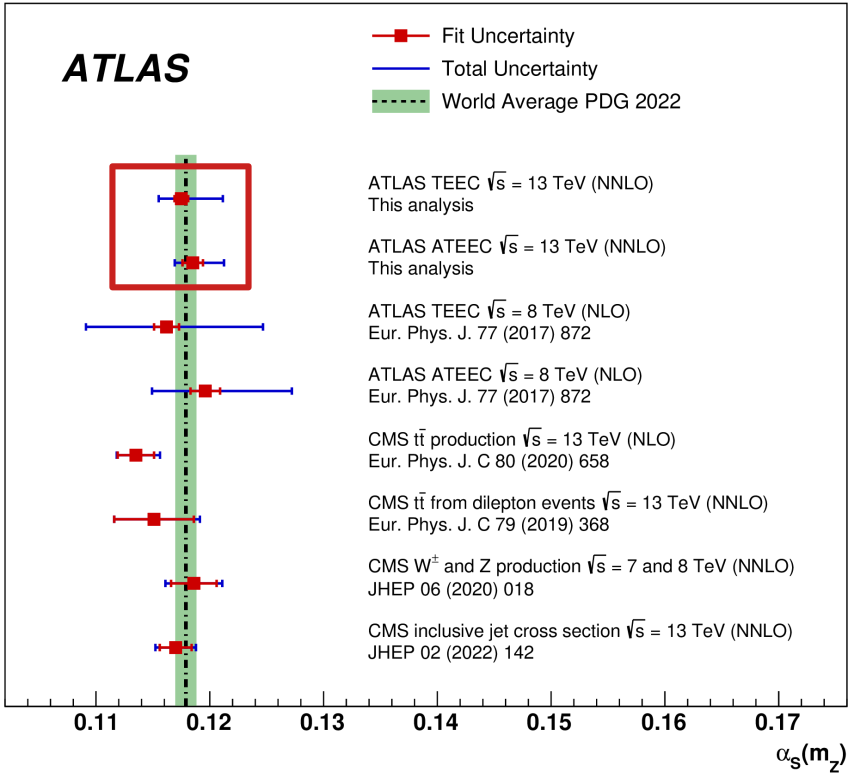
For comparison:
scale dependence (dominant theory uncertainty)

- TEEC ($H_{T,2} > 1$ TeV) : ~2%
- Thrust : ~3-5 %

***O*(1%) sensitivity**

α_S from TEEC @ NNLO by ATLAS

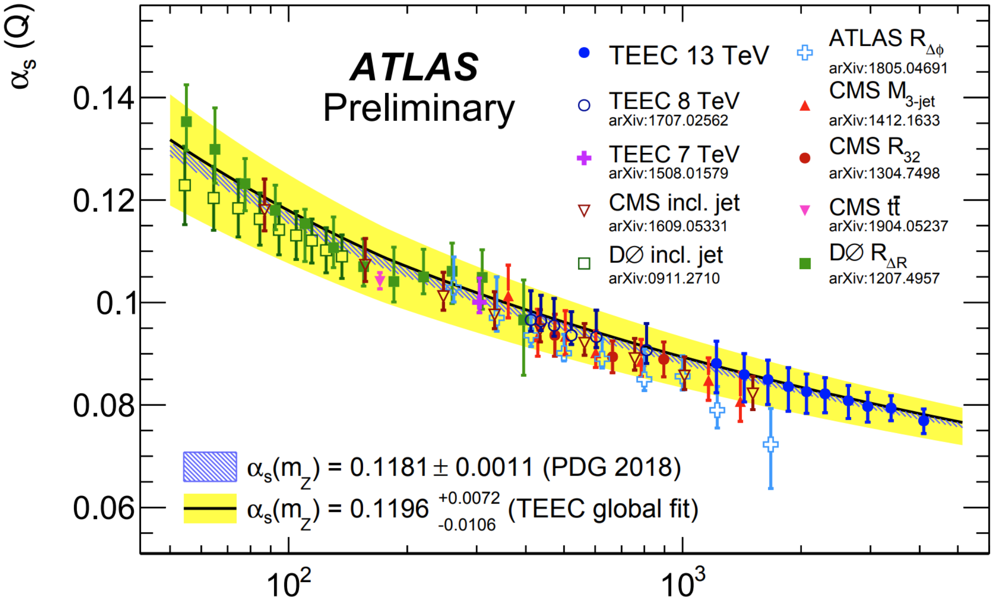
[ATLAS 2301.09351]



- NNLO QCD extraction from multi-jets → will contribute to the PDG average for the first time.
- Significant improvement to 8 TeV result mainly driven by NNLO QCD corrections
- Individual precision comparable to other measurements which include DIS and top or jets-data.

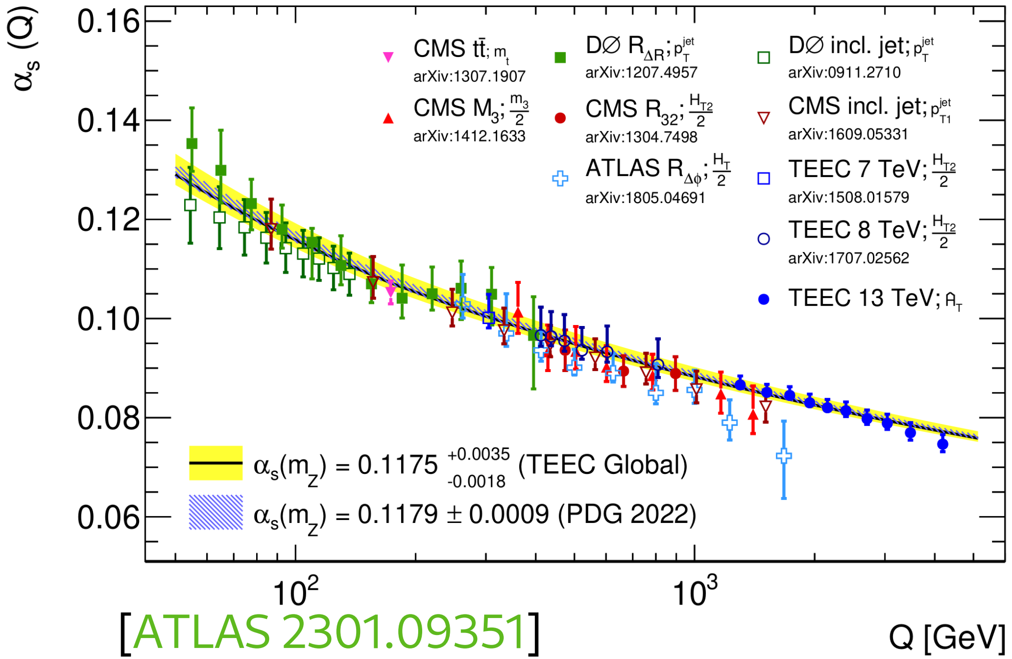
Running of α_S

NLO QCD



[ATLAS-CONF-2020-025]

NNLO QCD



[ATLAS 2301.09351]

Using the running of α_S to probe NP

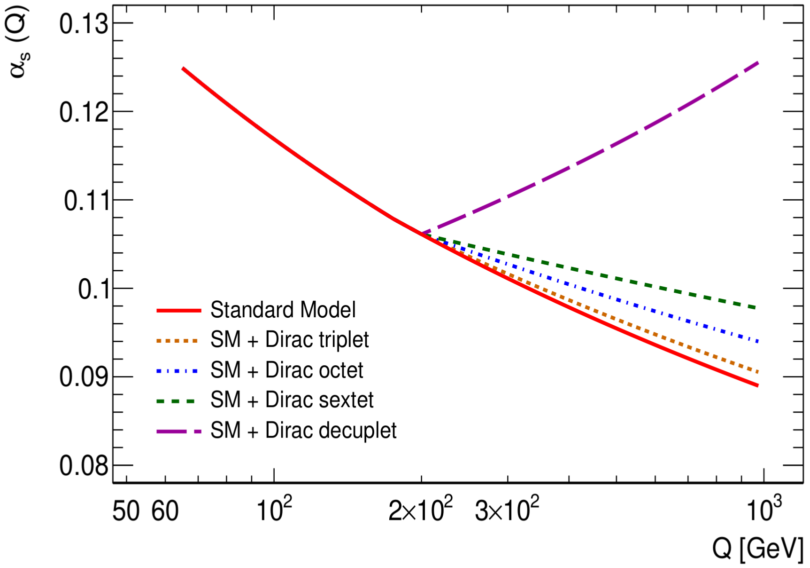
[Llorente, Nachman 1807.00894]

Indirect constraints to NP through modified running:

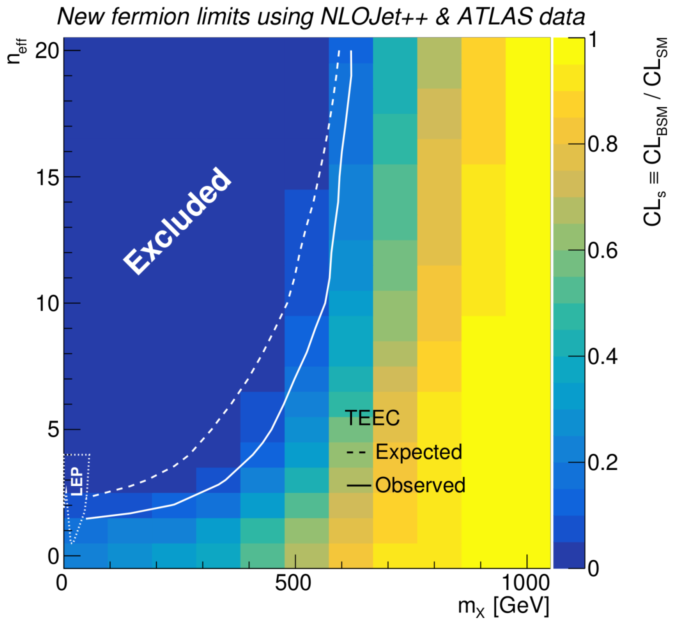
$$\beta_0 = \frac{1}{4\pi}\left(11 - \frac{2}{3}n_f - \frac{4}{3}n_X T_X\right)$$

$$\beta_1 = \frac{1}{(4\pi)^2}\left[102 - \frac{38}{3}n_f - 20 n_X T_X\left(1 + \frac{C_X}{5}\right)\right]$$

$$\alpha_s(Q) = \frac{1}{\beta_0 \log z}\left[1 - \frac{\beta_1}{\beta_0^2}\frac{\log(\log z)}{\log z}\right]; \quad z = \frac{Q^2}{\Lambda^2_{\mathrm{QCD}}}$$

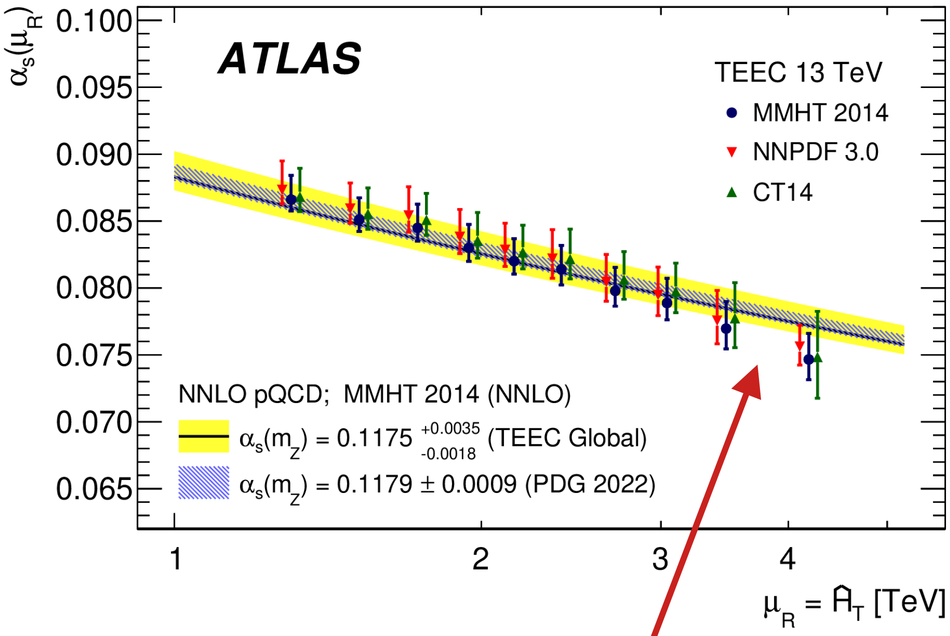


ATLAS TEEC @ 7 TeV data



Update with TEEC@13 TeV
→ much improved bounds

… or ‘new’ SM dynamics



Systematic slope
→ New physics?

Possible SM explanations

- Residual PDF effects → very high Q^2 ?
- EW corrections?
- Maybe effect from LC approximation in two-loop ME?

$$\mathcal{R}^{(2)}(\mu_R^2) = 2\,\mathrm{Re}\left[\mathcal{M}^{\dagger(0)}\mathcal{F}^{(2)}\right](\mu_R^2) + \left|\mathcal{F}^{(1)}\right|^2(\mu_R^2)$$

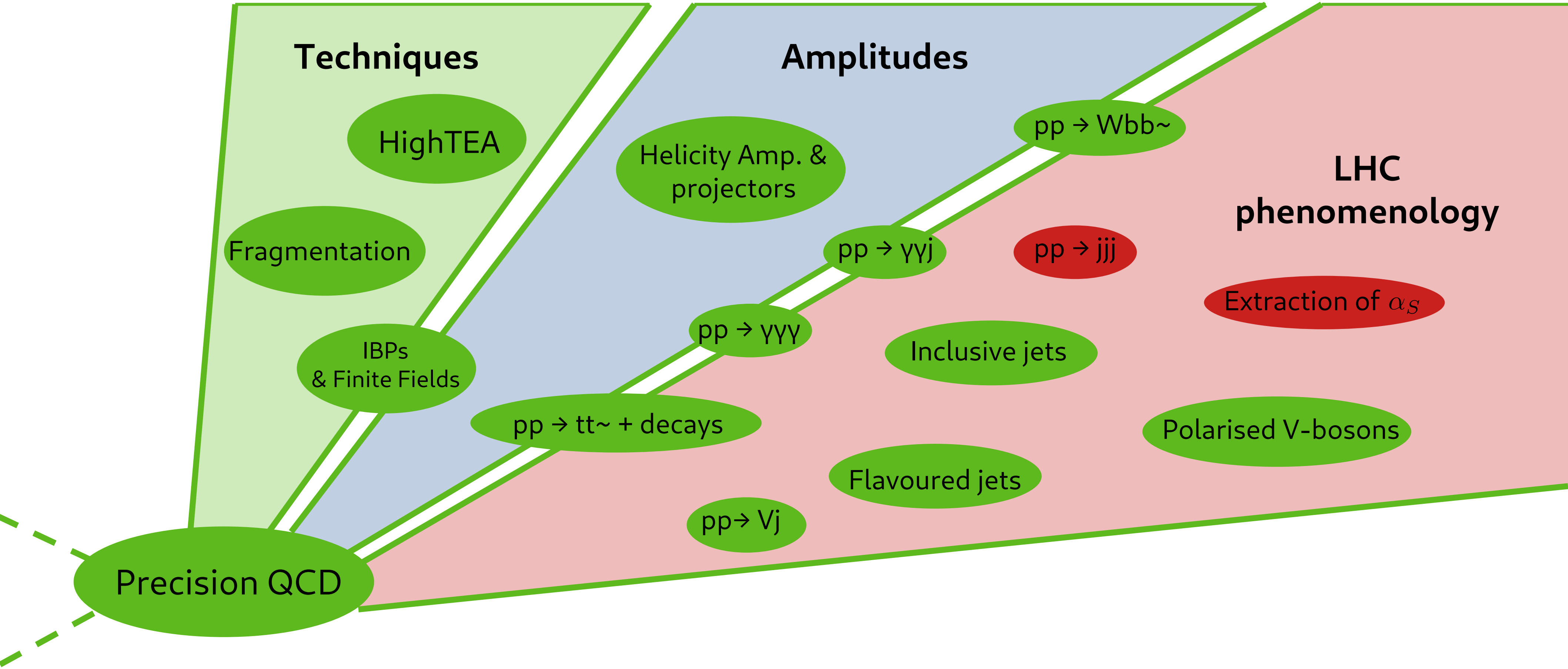
$$\equiv \mathcal{R}^{(2)}(s_{12}) + \sum_{i=1}^{4} c_i \ln^i\left(\frac{\mu_R^2}{s_{12}}\right)$$

$$\mathcal{R}^{(2)}(s_{12}) \approx \mathcal{R}^{(2)l.c.}(s_{12})$$

- Experimental systematics?
- Resummation?

Either case interesting!

The wider context



LHC Phenomenology

NNLO QCD computations

- **Top-quark pair production and leptonic decays**
  [1901.05407] [2008.11133]
  + b-quark fragmentation: [2102.08267] [2210.06078]
- **Vector + jets**
  W + charm-jet [2011.01011] [2212.00467]
  Z + b-jet [2205.11879]
- **Polarised vector-bosons**
  WW [2102.13583]
  W+jet [2109.14336] [2204.12394]
- **Inclusive jets**
  [1907.12911]
- **"2 → 3" processes**
  Three-photons [1911.00479]
  Diphoton+jet [2105.06940]
  Three jets [2106.05331] [2301.01086]
  W + 2 b-jets [2205.01687] [2209.03280]

Collaboration network

Main Collaborators:

- Michal Czakon (Aachen)
- Alexander Mitov (Cambridge)

Phenomenology:

- Mathieu Pellen (Freiburg)

Amplitudes:

- Herschel Chawdhry (Oxford)
- Bayu Hartanto (Cambridge), Simon Badger's group (Turin)
- Andreas van Hameren (Cracow)

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Exp. collaborations

DESY CMS top-quark group
(Behnke, Aldalya Martin)
→ [CMS-PAS-TOP-20-006]

Top spin-correlations in ATLAS
(Howard) → 1903.07570

W+charm CMS measurement
(Herandez) → to-appear-soon

Proposed COST network
COMETA

ATLAS multi-jet group at CERN
(Llorente, Roloff, LeBlanc)
→ α_S from TEEC 2301.09351
→ More to appear

Amplitudes

Two-loop amplitudes

- **pp → t t~**
  Polarised / Spin-Density-Matrix [1712.08075]
- **pp → γγγ (planar)**
  Squared matrix element [1911.00479]
  Helicity amplitudes [2012.13553]
- **pp → γγj (planar)**
  Helicity amplitudes [2103.04319]
- **pp → Wbb~ (planar)**
  Squared matrix element/SDM [2205.01687]
- **Non-planar five-point amplitudes**
  → work-in-progress

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Spectrum of techniques:

→ Tensor-reduction with projectors
→ Analytical IBP tables (IDSolver, IBPeasy)
→ Numerical DEQ for master integrals
→ Finite Field reconstructions (FiniteFlow and FireFly)

NNLO QCD techniques

Perturbative fragmentation @ NNLO QCD

- Proof-of-principle: [2102.08267]
- First fit of b-fragmentation functions: [2210.06078]

HighTEA

- Cloud service:
  "NNLO QCD ntuples" + user-friendly analysis
  → re-weighting + re-binning
- Prototype online, publication soon [230y.xxxxx]
  https://www.precision.hep.phy.cam.ac.uk/hightea/

Collaboration network

Michal Czakon (Aachen)
Alexander Mitov (Cambridge)
Terry Generet (Aachen)

Michal Czakon (Aachen)
Alexander Mitov (Cambridge)
Zahari Kassabov (Cambridge)

Future directions

Modern MC integration/sampling

- Improving performance of MC integration
- 1) "Nested sampling" → phase space explorations
  2) "Normalising flows" → phase space sampler

NNLO with massive bosons: B + 2-jet, BB + 1-jet

- A lot to do: amplitudes + cross sections
  But rich phenomenology!

Subtraction + Slicing: N3LO for 2 → 2 processes

- Ultimate precision for Drell-Yan, di-photon production, …

Collaboration network

Steffen Schumann (Göttingen)
David Yallup (Cambridge)

Michal Czakon (Aachen)
Alexander Mitov (Cambridge)
Bayu Hartanto (Cambridge)

Michal Czakon (Aachen)
Alexander Mitov (Cambridge)
Terry Generet (Aachen)

Summary & Outlook

Summary

- Three jet NNLO QCD predictions allow for precision phenomenology with multi-jet final states
- First predictions for R32 ratios and event shapes
- Extraction of the strong coupling constant from event shapes by ATLAS → will contribute to PDG ave.
- Relatively costly enterprise
  → effective NNLO QCD cross section tools needed
  → optimized STRIPPER subtraction scheme

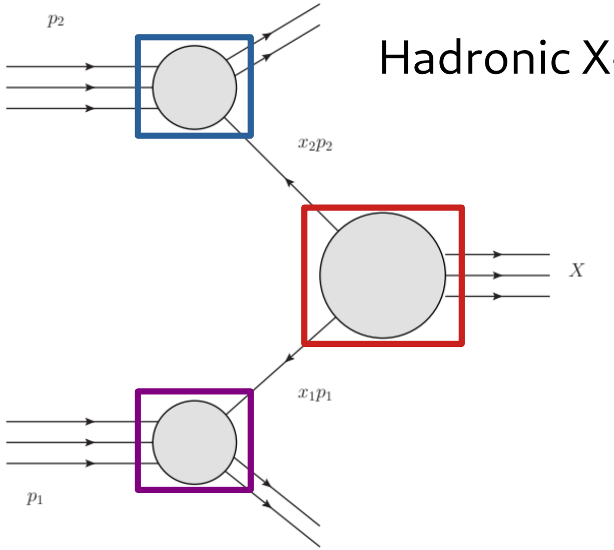
Outlook

- Many more observables are accessible: azimuthal decorrelation, earth-mover distance, …
- Still improvements to be made on subtractions schemes:
  - Better MC integration techniques → ML community has developed a plethora of tools
  - Technical aspects like form of selector function and phase space mappings

  "3 factors of 2 are also a order of magnitude" → difference between "doable" and "not doable"!

Backup

Hadronic cross section



Hadronic X-section: $\sigma_{h_1h_2\to X} = \sum_{ij}\int_0^1\int_0^1 \mathrm{d}x_1\mathrm{d}x_2\phi_{i,h_1}(x_1,\mu_F^2)\phi_{j/h_2}(x_2,\mu_F^2)\hat{\sigma}_{ij\to X}(\alpha_s(\mu_R^2),\mu_R^2,\mu_F^2)$

Parton distribution functions

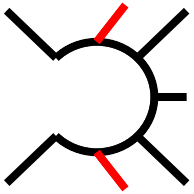
Perturbative expansion of partonic cross section:

$$\hat{\sigma}_{ab\to X} = \hat{\sigma}^{(0)}_{ab\to X} + \hat{\sigma}^{(1)}_{ab\to X} + \hat{\sigma}^{(2)}_{ab\to X} + \mathcal{O}(\alpha_s^3)$$

The NNLO bit: $\hat{\sigma}^{(2)}_{ab} = \hat{\sigma}^{\mathrm{RR}}_{ab} + \hat{\sigma}^{\mathrm{RV}}_{ab} + \hat{\sigma}^{\mathrm{VV}}_{ab} + \hat{\sigma}^{\mathrm{C2}}_{ab} + \hat{\sigma}^{\mathrm{C1}}_{ab}$

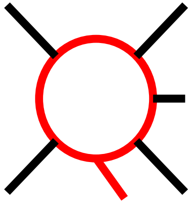
Double real radiation

$$\hat{\sigma}^{\mathrm{RR}}_{ab} = \frac{1}{2\hat{s}}\int \mathrm{d}\Phi_{n+2}\left\langle \mathcal{M}^{(0)}_{n+2}\middle|\mathcal{M}^{(0)}_{n+2}\right\rangle \mathrm{F}_{n+2}$$



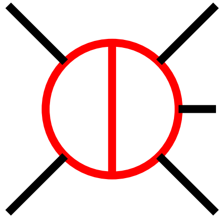
Real/Virtual correction

$$\hat{\sigma}^{\mathrm{RV}}_{ab} = \frac{1}{2\hat{s}}\int \mathrm{d}\Phi_{n+1}\, 2\mathrm{Re}\left\langle \mathcal{M}^{(0)}_{n+1}\middle|\mathcal{M}^{(1)}_{n+1}\right\rangle \mathrm{F}_{n+1}$$



Double virtual corrections

$$\hat{\sigma}^{\mathrm{VV}}_{ab} = \frac{1}{2\hat{s}}\int \mathrm{d}\Phi_{n}\left(2\mathrm{Re}\left\langle \mathcal{M}^{(0)}_{n}\middle|\mathcal{M}^{(2)}_{n}\right\rangle + \left\langle \mathcal{M}^{(1)}_{n}\middle|\mathcal{M}^{(1)}_{n}\right\rangle\right) \mathrm{F}_{n}$$



Partonic cross section beyond LO

Perturbative expansion of partonic cross section:

$$\hat{\sigma}_{ab\to X} = \hat{\sigma}^{(0)}_{ab\to X} + \hat{\sigma}^{(1)}_{ab\to X} + \hat{\sigma}^{(2)}_{ab\to X} + \mathcal{O}(\alpha_s^3)$$

Contributions with different multiplicities and # convolutions:

$$\hat{\sigma}^{(2)}_{ab} = \underline{\hat{\sigma}^{\mathrm{RR}}_{ab} + \hat{\sigma}^{\mathrm{RV}}_{ab} + \hat{\sigma}^{\mathrm{VV}}_{ab} + \hat{\sigma}^{\mathrm{C2}}_{ab} + \hat{\sigma}^{\mathrm{C1}}_{ab}}$$

$$\hat{\sigma}^{\mathrm{RR}}_{ab} = \frac{1}{2\hat{s}} \int \mathrm{d}\Phi_{n+2} \left\langle \mathcal{M}^{(0)}_{n+2} \middle| \mathcal{M}^{(0)}_{n+2} \right\rangle \mathrm{F}_{n+2}$$

$$\hat{\sigma}^{\mathrm{RV}}_{ab} = \frac{1}{2\hat{s}} \int \mathrm{d}\Phi_{n+1}\, 2\mathrm{Re} \left\langle \mathcal{M}^{(0)}_{n+1} \middle| \mathcal{M}^{(1)}_{n+1} \right\rangle \mathrm{F}_{n+1}$$

$$\hat{\sigma}^{\mathrm{VV}}_{ab} = \frac{1}{2\hat{s}} \int \mathrm{d}\Phi_{n} \left(2\mathrm{Re} \left\langle \mathcal{M}^{(0)}_{n} \middle| \mathcal{M}^{(2)}_{n} \right\rangle + \left\langle \mathcal{M}^{(1)}_{n} \middle| \mathcal{M}^{(1)}_{n} \right\rangle \right) \mathrm{F}_{n}$$

$$\hat{\sigma}^{\mathrm{C1}}_{ab} = (\text{single convolution})\, \mathrm{F}_{n+1}$$

$$\hat{\sigma}^{\mathrm{C2}}_{ab} = (\text{double convolution})\, \mathrm{F}_{n}$$

Each term separately IR divergent. But sum is:

→ finite

→ regularization scheme independent

Considering CDR ($d = 4 - 2\epsilon$):

→ Laurent expansion:

$$\hat{\sigma}^{C}_{ab} = \sum_{i=-4}^{0} c_i \epsilon^i + \mathcal{O}(\epsilon)$$

Sector decomposition I

Considering working in CDR:

→ Virtuals are usually done in this regularization

→ Real radiation:

→ Very difficult integrals, analytical impractical (except very simple cases)!

→ Numerics not possible, integrals are divergent: ε-poles!

How to extract these poles? → Sector decomposition!

Divide and conquer the phase space:

$$1 = \sum_{i,j}\left[\sum_k \mathcal{S}_{ij,k} + \sum_{k,l}\mathcal{S}_{i,k;j,l}\right] \quad \Rightarrow \quad \hat{\sigma}^{\mathrm{RR}}_{ab} = \frac{1}{2\hat{s}}\int \mathrm{d}\Phi_{n+2}\sum_{i,j}\left[\sum_k \mathcal{S}_{ij,k} + \sum_{k,l}\mathcal{S}_{i,k;j,l}\right]\left\langle \mathcal{M}^{(0)}_{n+2}\middle|\mathcal{M}^{(0)}_{n+2}\right\rangle \mathrm{F}_{n+2}$$

Sector decomposition II

Divide and conquer the phase space:

→ Each $\mathcal{S}_{ij,k}/\mathcal{S}_{i,k;j,l}$ has simpler divergences.

appearing as $1/s_{ijk} \quad 1/s_{ik}/s_{jl}$

Soft and collinear (w.r.t parton k,l) of partons i and j

→ Parametrization w.r.t. reference parton:

$$\hat{\eta}_i = \frac{1}{2}(1-\cos\theta_{ir}) \in [0,1] \qquad \hat{\xi}_i = \frac{u_i^0}{u_{\max}^0} \in [0,1]$$

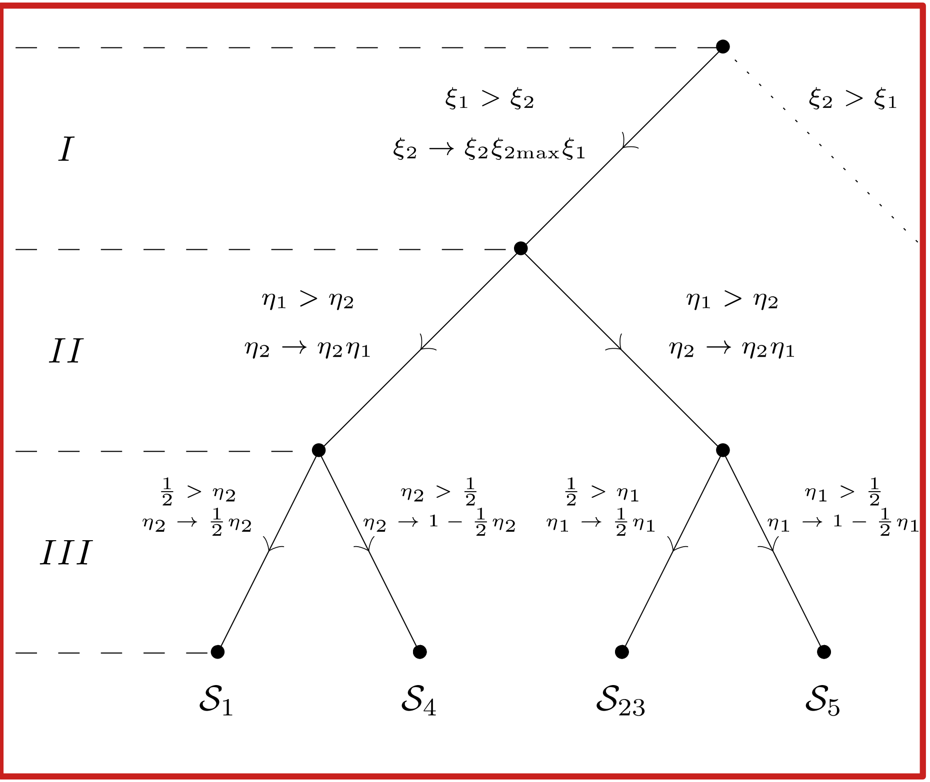
→ Subdivide to factorize divergences

$$s_{u_1u_2k} = (p_k + u_1 + u_2)^2 \sim \hat{\eta}_1 u_1^0 + \hat{\eta}_2 u_2^0 + \hat{\eta}_3 u_1^0 u_2^0$$

→ double soft factorization:

$$\theta(u_1^0 - u_2^0) + \theta(u_2^0 - u_1^0)$$

→ triple collinear factorization



[Czakon'10,Caola'17]

Sector decomposition III

Factorized singular limits in each sector:

$$\frac{1}{2\hat{s}}\int \mathrm{d}\Phi_{n+2}\,\mathcal{S}_{kl,m}\left\langle \mathcal{M}^{(0)}_{n+2}\middle|\mathcal{M}^{(0)}_{n+2}\right\rangle \mathrm{F}_{n+2} = \sum_{\text{sub-sec.}}\int \mathrm{d}\Phi_n \prod \mathrm{d}x_i\, \underbrace{x_i^{-1-b_i\epsilon}}_{\text{singular}}\,\mathrm{d}\tilde{\mu}(\{x_i\}) \underbrace{\prod x_i^{a_i+1}\,\langle \mathcal{M}_{n+2}|\mathcal{M}_{n+2}\rangle}_{\text{regular}}\,\mathrm{F}_{n+2}$$

Regularization of divergences:

$$x^{-1-b\epsilon} = \underbrace{\frac{-1}{b\epsilon}}_{\text{pole term}} + \underbrace{\left[x^{-1-b\epsilon}\right]_+}_{\text{reg. + sub.}}$$

$$\int_0^1 \mathrm{d}x\,\left[x^{-1-b\epsilon}\right]_+ f(x) = \int_0^1 \frac{f(x)-f(0)}{x^{1+b\epsilon}}$$

Finite NNLO cross section

$$\hat{\sigma}_{ab}^{\mathrm{RR}} = \frac{1}{2\hat{s}} \int \mathrm{d}\Phi_{n+2} \left\langle \mathcal{M}_{n+2}^{(0)} \middle| \mathcal{M}_{n+2}^{(0)} \right\rangle \mathrm{F}_{n+2}$$

$$\hat{\sigma}_{ab}^{\mathrm{RV}} = \frac{1}{2\hat{s}} \int \mathrm{d}\Phi_{n+1}\, 2\mathrm{Re} \left\langle \mathcal{M}_{n+1}^{(0)} \middle| \mathcal{M}_{n+1}^{(1)} \right\rangle \mathrm{F}_{n+1}$$

$$\hat{\sigma}_{ab}^{\mathrm{VV}} = \frac{1}{2\hat{s}} \int \mathrm{d}\Phi_{n} \left(2\mathrm{Re} \left\langle \mathcal{M}_{n}^{(0)} \middle| \mathcal{M}_{n}^{(2)} \right\rangle + \left\langle \mathcal{M}_{n}^{(1)} \middle| \mathcal{M}_{n}^{(1)} \right\rangle \right) \mathrm{F}_{n}$$

$$\hat{\sigma}_{ab}^{\mathrm{C1}} = (\text{single convolution})\, \mathrm{F}_{n+1}$$

$$\hat{\sigma}_{ab}^{\mathrm{C2}} = (\text{double convolution})\, \mathrm{F}_{n}$$

sector decomposition and master formula

$$x^{-1-b\epsilon} = \underbrace{\frac{-1}{b\epsilon}}_{\text{pole term}} + \underbrace{\left[x^{-1-b\epsilon}\right]_+}_{\text{reg. + sub.}}$$

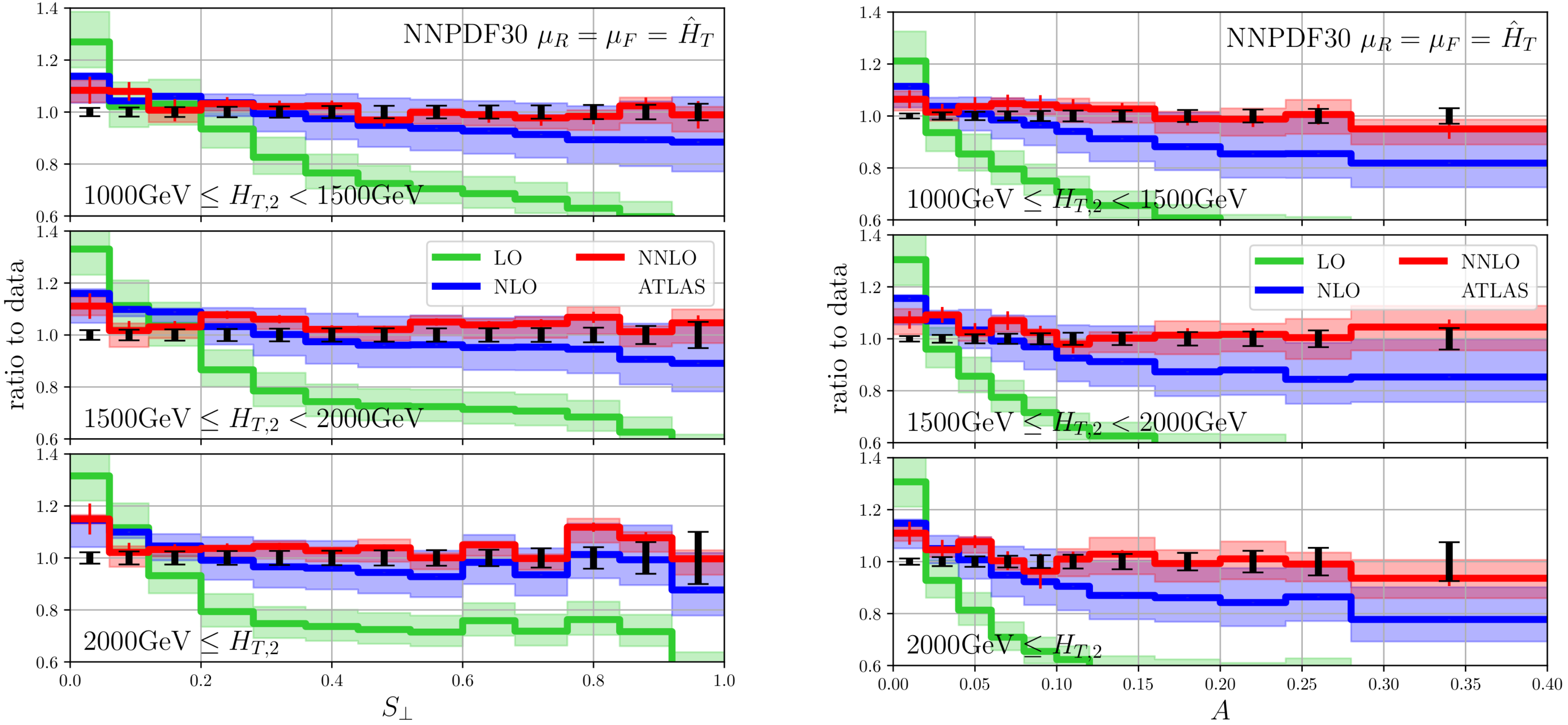
$$\left(\sigma_F^{RR}, \sigma_{SU}^{RR}, \sigma_{DU}^{RR}\right) \quad \left(\sigma_F^{RV}, \sigma_{SU}^{RV}, \sigma_{DU}^{RV}\right) \quad \left(\sigma_F^{VV}, \sigma_{DU}^{VV}, \sigma_{FR}^{VV}\right) \quad \left(\sigma_{SU}^{C1}, \sigma_{DU}^{C1}\right) \quad \left(\sigma_{DU}^{C2}, \sigma_{FR}^{C2}\right)$$

re-arrangement of terms → 4-dim. formulation [Czakon'14, Czakon'19]

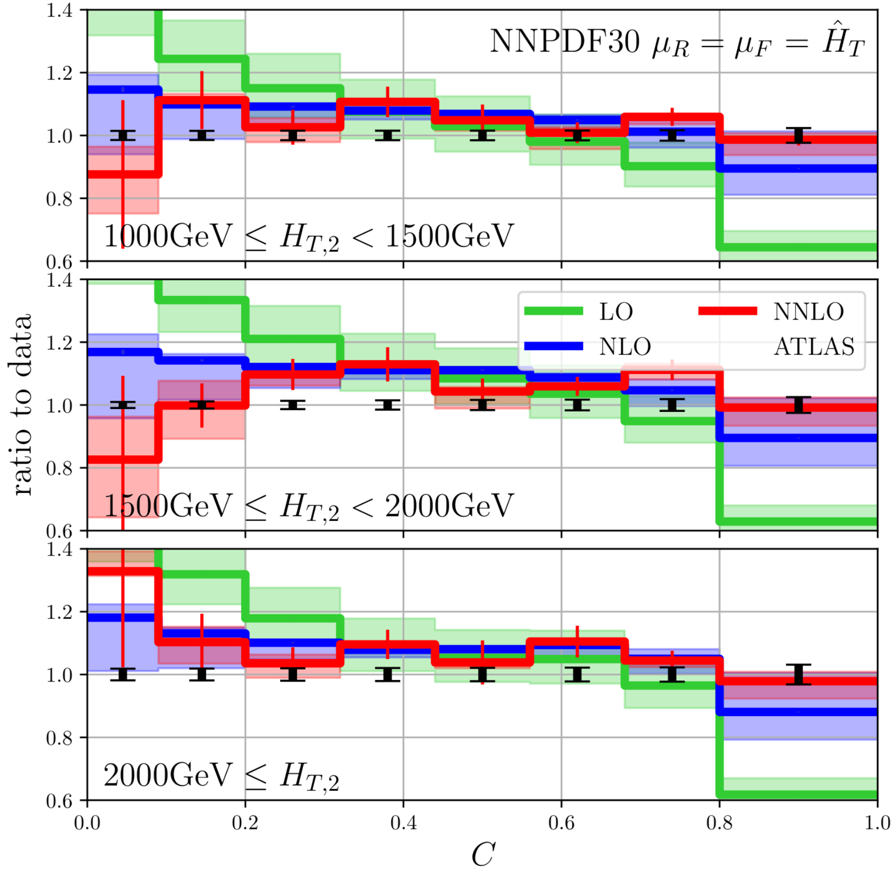
$$\underline{\left(\sigma_F^{RR}\right)} \quad \underline{\left(\sigma_F^{RV}\right)} \quad \underline{\left(\sigma_F^{VV}\right)} \quad \underline{\left(\sigma_{SU}^{RR}, \sigma_{SU}^{RV}, \sigma_{SU}^{C1}\right)} \quad \underline{\left(\sigma_{DU}^{RR}, \sigma_{DU}^{RV}, \sigma_{DU}^{VV}, \sigma_{DU}^{C1}, \sigma_{DU}^{C2}\right)} \quad \underline{\left(\sigma_{FR}^{RV}, \sigma_{FR}^{VV}, \sigma_{FR}^{C2}\right)}$$

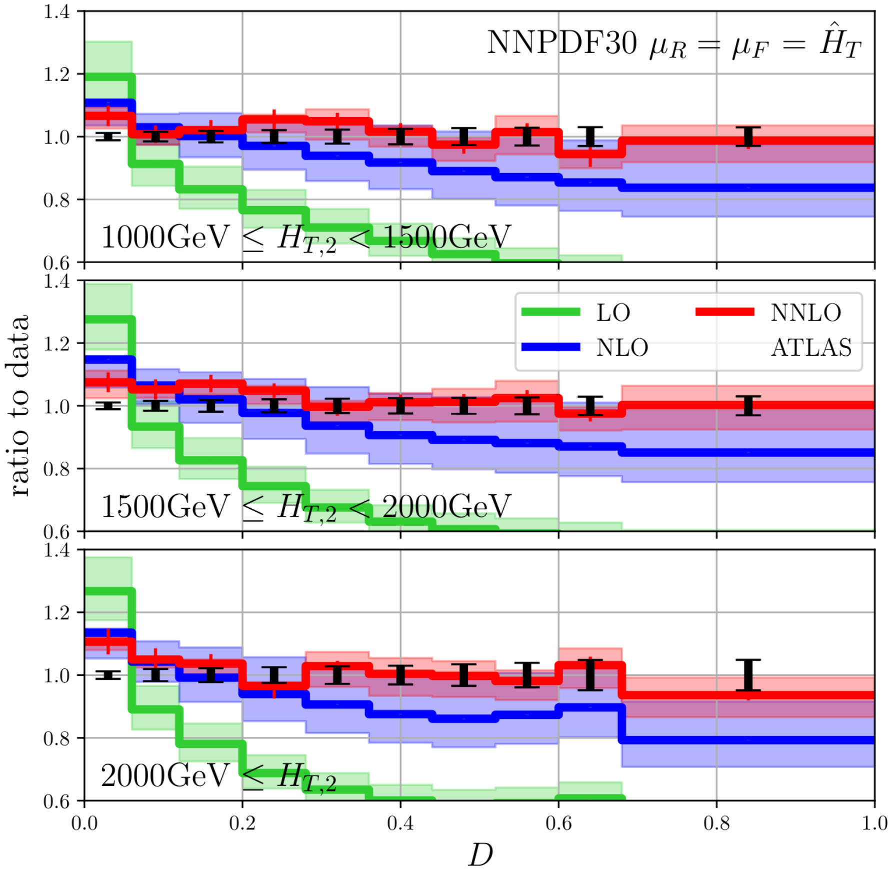
separately finite: ε poles cancel

More event-shapes I



More event-shapes II





Event shapes as MC tuning tool