IFJ PAN
Theory Division – Particle Theory
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QUANTUM FIELD THEORY

EXERCISES 1

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1 Canonical Quantization

1. *Klein-Gordon Hamiltonian*

   (a) Show, starting from the Lagrangian for a single Klein-Gordon field,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi(x))^2 - \frac{1}{2}m^2(\phi(x))^2 \,, \tag{1.1}$$

   that the Hamiltonian (density), expressed in the canonical variables $\phi(x)$ and $\pi(x) = \partial_t \phi(x)$, can be written as

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 \,. \tag{1.2}$$

   (b) Demonstrate, using the expression of the canonical variables ϕ and π in terms of creation ($a^\dagger_{\vec{p}}$) and annihilation ($a_{\vec{p}}$) operators, that:

$$H = \int \mathrm{d}^3x \mathcal{H} = \int \frac{\mathrm{d}^3p}{(2\pi)^3} E_{\vec{p}} \left(a^\dagger_{\vec{p}} a_{\vec{p}} + \frac{1}{2}\left[a_{\vec{p}}, a^\dagger_{\vec{p}}\right] \right) . \tag{1.3}$$

   For reference:

$$\phi(\vec{x}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}} + a^\dagger_{-\vec{p}}\right) e^{i\vec{p}\vec{x}} \tag{1.4}$$

$$\pi(\vec{x}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3} (-i) \sqrt{\frac{E_{\vec{p}}}{2}} \left(a_{\vec{p}} - a^\dagger_{-\vec{p}}\right) e^{i\vec{p}\vec{x}} \tag{1.5}$$

2. *Feynman propagator*

   Consider the quantity

$$D(x-y) = \langle 0| \phi(x)\phi(y) |0\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{-ip(x-y)} \,. \tag{1.6}$$

Show that

$$\langle 0| \left[\phi(x), \phi(y)\right] |0\rangle = D(x-y) - D(y-x) \tag{1.7}$$

and under the assumption that $x^0 > y^0$:

$$\langle 0| \left[\phi(x), \phi(y)\right] |0\rangle = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip(x-y)} \tag{1.8}$$

Hint: use the residue theorem for a integral of an analytic function $f(z)$ along a curve C encircling simple poles c_i

$$\oint_C f(z)\mathrm{d}z = 2\pi i \sum_i \mathrm{Res}(f, c_i)\ , \tag{1.9}$$

and the rule of L'Hopital for $f(z) = \frac{g(z)}{h(z)}$ with $h(c) = 0$ but $h'(c) \neq 0$,

$$\mathrm{Res}(f, c) = \frac{g(c)}{h'(c)}\ . \tag{1.10}$$